Massless On-shell Double Box

```
(* Time of evaluation and the version of the Mathematica kernel used *)

DateString[]

Sat 26 Sep 2015 15:47:06

$Version

9.0 for Linux x86 (64-bit) (November 20, 2012)
```

Derivation of MB representation

```
(* On the basis of http://prac.us.edu.pl/~gluza/ambre/examples/example7.nb *)

<< MB.m

MB 1.2

by Michal Czakon

improvements by Alexander Smirnov

more info in hep-ph/0511200

last modified 2 Jan 09

<< AMBREv1.2.m

by K.Kajda   ver: 1.2
last modified 9 Apr 2008
last executed on 26.09.2015 at 15:47

(* barnesroutines.m by David A.Kosower, https://mbtools.hepforge.org/ *)

<< barnesroutines.m

Barnes Routines, v 1.1.1 of July 23, 2009

m = 0

0

B1 = Fullintegral[
   {1},
   {PR[k1, m, n1] PR[k1 + p1, 0, n2] PR[k1 + p1 + p2, m, n3] PR[k1 - k2, 0, n4]
     PR[k2, m, n5] PR[k2 + p1 + p2, m, n6] PR[k2 - p3, 0, n7]}, {k2, k1}];

invariants = {p1^2 → m^2, p2^2 → m^2, p3^2 → m^2, p4^2 → m^2, p1 * p2 → 1 / 2 * s - m^2,
    p3 * p4 → 1 / 2 s - m^2, p1 * p3 → 1 / 2 * t - m^2, p2 * p4 → 1 / 2 * t - m^2,
    p2 * p3 → 1 / 2 * u - m^2, p1 * p4 → 1 / 2 * u - m^2} /. u → 4 m^2 - s - t // Expand;
```

IntPart[1]

```
numerator=1
integral=PR[k1 - k2, 0, n4] PR[k2, 0, n5] PR[k2 + p1 + p2, 0, n6] PR[k2 - p3, 0, n7]
momentum=k2
```

Fauto::mode :
U and F polynomials will be calculated in AUTO mode. In order to use MANUAL mode execute Fauto[0].

SubLoop[integral]

```
Iteration nr1: >>Integrating over k2<<

Computing U & F polynomial in AUTO mode >>Fauto[1]<<

U polynomial...
X[1] + X[2] + X[3] + X[4]
F polynomial...
- PR[k1, 0] X[1] X[2] - PR[k1 + p1 + p2, 0] X[1] X[3] - s X[2] X[3] - PR[k1 - p3, 0] X[1] X[4]

Representation after integrating over: k2...

SubLoop1[((-1)^(2-eps-z3) (-s)^z3 Gamma[-z1] Gamma[-z2] Gamma[2 - eps - n5 - n6 - n7 - z3]
    Gamma[2 - eps - n4 - n5 - n6 - z1 - z2 - z3] Gamma[-z3] Gamma[n5 + z1 + z3]
    Gamma[n6 + z2 + z3] Gamma[-2 + eps + n4 + n5 + n6 + n7 + z1 + z2 + z3]) /
  (Gamma[n4] Gamma[n5] Gamma[n6] Gamma[4 - 2 eps - n4 - n5 - n6 - n7] Gamma[n7]),
 PR[k1, 0, z1] PR[k1 + p1 + p2, 0, z2] PR[k1 - p3, 0, 2 - eps - n4 - n5 - n6 - n7 - z1 - z2 - z3]]
```

IntPart[2]

```
numerator=1
integral=PR[k1, 0, n1 - z1] PR[k1 + p1, 0, n2]
  PR[k1 + p1 + p2, 0, n3 - z2] PR[k1 - p3, 0, -2 + eps + n4 + n5 + n6 + n7 + z1 + z2 + z3]
momentum=
 k1
```

Fauto::mode :
U and F polynomials will be calculated in AUTO mode. In order to use MANUAL mode execute Fauto[0].

repr = SubLoop[integral];

```
Iteration nr2: >>Integrating over k1<<

Computing U & F polynomial in AUTO mode >>Fauto[1]<<

U polynomial...
X[1] + X[2] + X[3] + X[4]
F polynomial...
- s X[1] X[3] - t X[2] X[4]

Final representation:

((-1)^(n1+n2+n3+n4+n5+n6+n7) (-s)^(z3+z4) (-t)^(4-2 eps-n1-n2-n3-n4-n5-n6-n7-z3-z4) Gamma[-z1] Gamma[-z2]
   Gamma[2 - eps - n5 - n6 - n7 - z3] Gamma[2 - eps - n4 - n5 - n6 - z1 - z2 - z3] Gamma[-z3]
   Gamma[n5 + z1 + z3] Gamma[n6 + z2 + z3] Gamma[2 - eps - n1 - n2 - n3 + z1 + z2 - z4]
   Gamma[4 - 2 eps - n1 - n3 - n4 - n5 - n6 - n7 - z3 - z4] Gamma[-z4] Gamma[n1 - z1 + z4]
   Gamma[n3 - z2 + z4] Gamma[-4 + 2 eps + n1 + n2 + n3 + n4 + n5 + n6 + n7 + z3 + z4]) /
 (Gamma[n2] Gamma[n4] Gamma[n5] Gamma[n6] Gamma[4 - 2 eps - n4 - n5 - n6 - n7] Gamma[n7]
   Gamma[n1 - z1] Gamma[n3 - z2] Gamma[6 - 3 eps - n1 - n2 - n3 - n4 - n5 - n6 - n7 - z3])
```

```
BarnesLemma[repr, 1]
```

```
>> Barnes 1st Lemma will be checked for: {z2, z1} <<
   Starting with dim=4 representation...

1. Checking z2
2. Checking z1
>> Representation after 1st Barnes Lemma:  <<
   Could not apply Barnes-Lemma
```

$$\Big((-1)^{n1+n2+n3+n4+n5+n6+n7}\,(-s)^{z3+z4}\,(-t)^{4-2\,eps-n1-n2-n3-n4-n5-n6-n7-z3-z4}\,\text{Gamma}[-z1]\,\text{Gamma}[-z2]\,\text{Gamma}[2-eps-n5-n6-n7-z3]\,\text{Gamma}[2-eps-n4-n5-n6-z1-z2-z3]\,\text{Gamma}[-z3]\,\text{Gamma}[n5+z1+z3]\,\text{Gamma}[n6+z2+z3]\,\text{Gamma}[2-eps-n1-n2-n3+z1+z2-z4]\,\text{Gamma}[4-2\,eps-n1-n3-n4-n5-n6-n7-z3-z4]\,\text{Gamma}[-z4]\,\text{Gamma}[n1-z1+z4]\,\text{Gamma}[n3-z2+z4]\,\text{Gamma}[-4+2\,eps+n1+n2+n3+n4+n5+n6+n7+z3+z4]\Big)\Big/\big(\text{Gamma}[n2]\,\text{Gamma}[n4]\,\text{Gamma}[n5]\,\text{Gamma}[n6]\,\text{Gamma}[4-2\,eps-n4-n5-n6-n7]\,\text{Gamma}[n7]\,\text{Gamma}[n1-z1]\,\text{Gamma}[n3-z2]\,\text{Gamma}[6-3\,eps-n1-n2-n3-n4-n5-n6-n7-z3]\big)$$

```
fin = % /. {n1 → 1, n2 → 1, n3 → 1, n4 → 1, n5 → 1, n6 → 1, n7 → 1}
```

$$-\Big((-s)^{z3+z4}\,(-t)^{-3-2\,eps-z3-z4}\,\text{Gamma}[-z1]\,\text{Gamma}[-z2]\,\text{Gamma}[-1-eps-z3]\,\text{Gamma}[-1-eps-z1-z2-z3]\,\text{Gamma}[-z3]\,\text{Gamma}[1+z1+z3]\,\text{Gamma}[1+z2+z3]\,\text{Gamma}[-1-eps+z1+z2-z4]\,\text{Gamma}[-2-2\,eps-z3-z4]\,\text{Gamma}[-z4]\,\text{Gamma}[1-z1+z4]\,\text{Gamma}[1-z2+z4]\,\text{Gamma}[3+2\,eps+z3+z4]\Big)\Big/\big(\text{Gamma}[-2\,eps]\,\text{Gamma}[1-z1]\,\text{Gamma}[1-z2]\,\text{Gamma}[-1-3\,eps-z3]\big)$$

```
Kfin = fin * (-s) ^ (2 + 2 * eps) * (-t)
```

$$\Big((-s)^{2+2\,eps+z3+z4}\,(-t)^{-3-2\,eps-z3-z4}\,t\,\text{Gamma}[-z1]\,\text{Gamma}[-z2]\,\text{Gamma}[-1-eps-z3]\,\text{Gamma}[-1-eps-z1-z2-z3]\,\text{Gamma}[-z3]\,\text{Gamma}[1+z1+z3]\,\text{Gamma}[1+z2+z3]\,\text{Gamma}[-1-eps+z1+z2-z4]\,\text{Gamma}[-2-2\,eps-z3-z4]\,\text{Gamma}[-z4]\,\text{Gamma}[1-z1+z4]\,\text{Gamma}[1-z2+z4]\,\text{Gamma}[3+2\,eps+z3+z4]\Big)\Big/\big(\text{Gamma}[-2\,eps]\,\text{Gamma}[1-z1]\,\text{Gamma}[1-z2]\,\text{Gamma}[-1-3\,eps-z3]\big)$$

```
Kfin = Kfin /. (-s)^(2+2 eps+z3+z4) (-t)^(-3-2 eps-z3-z4) t → - (t / s)^-(2+2 eps+z3+z4)
```

$$-\Big(\left(\frac{t}{s}\right)^{-2-2\,eps-z3-z4}\,\text{Gamma}[-z1]\,\text{Gamma}[-z2]\,\text{Gamma}[-1-eps-z3]\,\text{Gamma}[-1-eps-z1-z2-z3]\,\text{Gamma}[-z3]\,\text{Gamma}[1+z1+z3]\,\text{Gamma}[1+z2+z3]\,\text{Gamma}[-1-eps+z1+z2-z4]\,\text{Gamma}[-2-2\,eps-z3-z4]\,\text{Gamma}[-z4]\,\text{Gamma}[1-z1+z4]\,\text{Gamma}[1-z2+z4]\,\text{Gamma}[3+2\,eps+z3+z4]\Big)\Big/\big(\text{Gamma}[-2\,eps]\,\text{Gamma}[1-z1]\,\text{Gamma}[1-z2]\,\text{Gamma}[-1-3\,eps-z3]\big)$$

```
Kfin = Kfin /. t / s → x
```

$$-\Big(x^{-2-2\,eps-z3-z4}\,\text{Gamma}[-z1]\,\text{Gamma}[-z2]\,\text{Gamma}[-1-eps-z3]\,\text{Gamma}[-1-eps-z1-z2-z3]\,\text{Gamma}[-z3]\,\text{Gamma}[1+z1+z3]\,\text{Gamma}[1+z2+z3]\,\text{Gamma}[-1-eps+z1+z2-z4]\,\text{Gamma}[-2-2\,eps-z3-z4]\,\text{Gamma}[-z4]\,\text{Gamma}[1-z1+z4]\,\text{Gamma}[1-z2+z4]\,\text{Gamma}[3+2\,eps+z3+z4]\Big)\Big/\big(\text{Gamma}[-2\,eps]\,\text{Gamma}[1-z1]\,\text{Gamma}[1-z2]\,\text{Gamma}[-1-3\,eps-z3]\big)$$

```
rules = MBoptimizedRules[Kfin, eps → 0, {}, {eps}]
```

MBresidues::contour : contour starts and/or ends on a pole of Gamma[−1 − 2 eps − z2 − z3]

MBresidues::contour : contour starts and/or ends on a pole of Gamma[−1 − eps + z1 + z2 − z4]

MBrules::norules : no rules could be found to regulate this integral

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General::stop : Further output of MBrules::norules will be suppressed during this calculation. ≫

$$\left\{\left\{\text{eps} \to -\frac{103}{128}\right\}, \left\{\text{z1} \to -\frac{15}{128}, \text{z2} \to -\frac{79}{128}, \text{z3} \to -\frac{41}{128}, \text{z4} \to -\frac{131}{128}\right\}\right\}$$

```
integrals = MBcontinue[Kfin, eps → 0, rules, Verbose → False];

ser = MBexpand[{integrals}, Exp[2 * eps EulerGamma],
   {eps, 0, 0}] // MBmerge
```

```
{MBint[1/(90 eps^4) (-360 + 210 eps^2 π^2 + 124 eps^4 π^4 + 15 eps^2 (-12 + 25 eps^2 π^2) Log[x]^2 -
     30 eps^3 Log[x]^3 + 75 eps^4 Log[x]^4 - 1065 eps^3 PolyGamma[2, 1] +
     75 eps Log[x] (6 - 5 eps^2 π^2 + 20 eps^3 PolyGamma[2, 1])), {{eps → 0}, {}}],
 MBint[1/(12 eps^2 Gamma[1 - z1]) x^-z1 Gamma[-z1]^2 Gamma[z1]
    (-6 eps Gamma[-z1] (4 eps Gamma[z1] (x^z1 + 2 Gamma[1 - z1] Gamma[1 + z1]) -
        x^z1 Gamma[1 + z1] (1 + 4 eps EulerGamma + 2 eps PolyGamma[0, 1 - z1] - 3 eps
             PolyGamma[0, -z1] + 3 eps PolyGamma[0, z1] + 2 eps PolyGamma[0, 1 + z1])) +
      x^z1 Gamma[1 - z1] Gamma[1 + z1] (-6 - 12 eps EulerGamma - 12 eps^2 EulerGamma^2 +
        4 eps^2 π^2 + 12 eps Log[x] + 24 eps^2 EulerGamma Log[x] + 9 eps^2 PolyGamma[0, -z1]^2 -
        6 eps (1 + 2 eps EulerGamma - 2 eps Log[x]) PolyGamma[0, z1] -
        3 eps^2 PolyGamma[0, z1]^2 + 24 eps^2 Log[x] PolyGamma[0, 1 + z1] +
        12 eps^2 PolyGamma[0, 1 + z1]^2 - 6 eps PolyGamma[0, -z1] (1 + 2 eps EulerGamma +
           2 eps Log[x] + eps PolyGamma[0, z1] + 4 eps PolyGamma[0, 1 + z1]) + 9 eps^2
         PolyGamma[1, -z1] + 21 eps^2 PolyGamma[1, z1] + 12 eps^2 PolyGamma[1, 1 + z1])),
  {{eps → 0}, {z1 → -15/128}}], MBint[-1/(12 eps^2 Gamma[1 - z2])
    x^-z2 Gamma[-z2]^2 Gamma[z2]
     (24 eps^2 Gamma[-z2] Gamma[z2] (x^z2 + 2 Gamma[1 - z2] Gamma[1 + z2]) +
       x^z2 Gamma[1 - z2] Gamma[1 + z2] (6 + 12 eps EulerGamma + 12 eps^2 EulerGamma^2 - 4 eps^2
           π^2 - 12 eps Log[x] - 24 eps^2 EulerGamma Log[x] - 9 eps^2 PolyGamma[0, -z2]^2 +
          6 eps (1 + 2 eps EulerGamma - 2 eps Log[x]) PolyGamma[0, z2] +
          3 eps^2 PolyGamma[0, z2]^2 - 24 eps^2 Log[x] PolyGamma[0, 1 + z2] -
          12 eps^2 PolyGamma[0, 1 + z2]^2 + 6 eps PolyGamma[0, -z2] (1 + 2 eps EulerGamma +
             2 eps Log[x] + eps PolyGamma[0, z2] + 4 eps PolyGamma[0, 1 + z2]) - 9 eps^2
           PolyGamma[1, -z2] - 21 eps^2 PolyGamma[1, z2] - 12 eps^2 PolyGamma[1, 1 + z2])),
  {{eps → 0}, {z2 → -79/128}}], MBint[1/eps 2 x^(-1-z3) Gamma[-1 - z3]^2
    Gamma[-z3]
```

```
Gamma[1 + z3]^2
Gamma[2 + z3]
(1 + eps EulerGamma - 3 eps Log[x] - 4 eps PolyGamma[0, -1 - z3] +
  2 eps PolyGamma[0, 1 + z3] + 3 eps PolyGamma[0, 2 + z3]),
{{eps -> 0}, {z3 -> -41/128}}], MBint[
-1/eps
 2 x^(-1-z4) Gamma[-1 - z4]^2 Gamma[-z4] Gamma[1 + z4]^2
  Gamma[2 + z4] (-1 + eps EulerGamma + eps Log[x] +
   2 eps PolyGamma[0, 1 + z4] - eps PolyGamma[0, 2 + z4]),
{{eps -> 0}, {z4 -> -131/128}}], MBint[-(Gamma[-z1] Gamma[z1 - z2] Gamma[-z2]
   (Gamma[1 - z1] Gamma[1 + z1] Gamma[-z2] Gamma[z2] +
    Gamma[-z1] Gamma[z1] Gamma[1 - z2] Gamma[1 + z2]) Gamma[-z1 + z2]) /
 (Gamma[1 - z1] Gamma[1 - z2]), {{eps -> 0}, {z1 -> -15/128,
 z2 -> -79/128}}],
MBint[-2 x^(-1-z1-z3) Gamma[-z1]
 Gamma[z1]
 Gamma[-1 - z1 - z3]^2
 Gamma[-z3]
 Gamma[1 + z3]
 Gamma[1 + z1 + z3]
 Gamma[2 + z1 + z3],
{{eps -> 0}, {z1 -> -15/128, z3 -> -41/128}}],
MBint[
 -2
 x^(-1-z2-z3)
 Gamma[-z2]
 Gamma[z2]
 Gamma[-1 - z2 - z3]^2
 Gamma[-z3]
 Gamma[1 + z3]
 Gamma[1 + z2 + z3]
 Gamma[2 + z2 + z3],
{{eps -> 0}, {z2 -> -79/128, z3 -> -41/128}}]}
```

Kmb = DoAllBarnes[ser, True] // MBmerge

```
{MBint[-4/eps^4 + (5 π^2)/(2 eps^2) + (EulerGamma^2 π^2)/4 + (923 π^4)/720 + (-2/eps^2 + (25 π^2)/6) Log[x]^2 -
    Log[x]^3/(3 eps) + (5 Log[x]^4)/6 - (65 PolyGamma[2, 1])/(6 eps) - 2 EulerGamma PolyGamma[2, 1] +
    Log[x] (5/eps^3 - (9 π^2)/(2 eps) + 41/3 PolyGamma[2, 1]), {{eps → 0}, {}}],
  MBint[-1/(4 Gamma[1 - z1]) x^-z1 Gamma[-z1]^2 Gamma[z1] Gamma[1 + z1]
      (-2 x^z1 Gamma[-z1] (2 PolyGamma[0, 1 - z1] - 3 PolyGamma[0, -z1] +
            3 PolyGamma[0, z1] + 2 PolyGamma[0, 1 + z1]) + Gamma[1 - z1]
          (16 Gamma[-z1] Gamma[z1] + x^z1 (PolyGamma[0, z1]^2 - 4 PolyGamma[0, 1 + z1]^2 -
                3 PolyGamma[1, -z1] - 7 PolyGamma[1, z1] - 4 PolyGamma[1, 1 + z1]))),
    {{eps → 0}, {z1 → -15/128}}], MBint[-1/4 x^-z2 Gamma[-z2]^2 Gamma[z2]
    Gamma[1 + z2]
    (16 Gamma[-z2] Gamma[z2] + x^z2 (PolyGamma[0, z2]^2 - 4 PolyGamma[0, 1 + z2]^2 -
          3 PolyGamma[1, -z2] - 7 PolyGamma[1, z2] - 4 PolyGamma[1, 1 + z2])),
    {{eps → 0}, {z2 → -79/128}}], MBint[Gamma[-z2] Gamma[z2]
    (Gamma[1 - z2] Gamma[z2] PolyGamma[1, 1 - z2] +
      Gamma[-z2] Gamma[1 + z2] PolyGamma[1, 1 + z2]), {{eps → 0}, {z2 → -1/2}}],
  MBint[-2 x^(-1-z3) Gamma[-1 - z3]^2 Gamma[-z3] Gamma[1 + z3]^2 Gamma[2 + z3]
    (EulerGamma + PolyGamma[0, -z3]),
    {{eps → 0}, {z3 → -15/16}}],
  MBint[-2 x^(-1-z3) Gamma[-1 - z3]^2 Gamma[-z3] Gamma[1 + z3]^2
    Gamma[2 + z3] (EulerGamma + PolyGamma[0, -z3]),
    {{eps → 0}, {z3 → -7/16}}],
  MBint[1/eps 2 x^(-1-z3) Gamma[-1 - z3]^2 Gamma[-z3] Gamma[1 + z3]^2 Gamma[2 + z3]
      (1 + eps EulerGamma - 3 eps Log[x] - 4 eps PolyGamma[0, -1 - z3] +
        2 eps PolyGamma[0, 1 + z3] + 3 eps PolyGamma[0, 2 + z3]), {{eps → 0}, {z3 → -41/128}}],
  MBint[-1/eps 2 x^(-1-z4) Gamma[-1 - z4]^2 Gamma[-z4] Gamma[1 + z4]^2 Gamma[2 + z4]
      (-1 + eps EulerGamma + eps Log[x] + 2 eps PolyGamma[0, 1 + z4] -
        eps PolyGamma[0, 2 + z4]), {{eps → 0}, {z4 → -131/128}}]}
```

```
(* Numerical check with V.A.Smirnov hep-ph/9905323 Eqs. 22-25 *)

Li[n_, z_] := PolyLog[n, z]
S[a_, b_, z_] := PolyLog[a, b, z]
```

```
K0t[x_] := -4 / eps^4 + 5 Log[x] / eps^3 - (2 Log[x]^2 - (5 / 2) Pi^2) / eps^2 -
  ((2 / 3) Log[x]^3 + 11 / 2 Pi^2 Log[x] - 65 / 3 Zeta[3]) / eps +
  4 / 3 Log[x]^4 + 6 Pi^2 Log[x]^2 - 88 / 3 Zeta[3] Log[x] + 29 / 30 Pi^4
```

```
K1t[x_] := - (2 Li[3, -x] - 2 Log[x] Li[2, -x] - (Log[x]^2 + Pi^2) Log[1 + x]) * 2 / eps -
  4 (S[2, 2, -x] - Log[x] S[1, 2, -x]) + 44 Li[4, -x] - 4 (Log[1 + x] + 6 Log[x]) Li[3, -x] +
  2 (Log[x]^2 + 2 Log[x] Log[1 + x] + 10 / 3 Pi^2) Li[2, -x] +
  (Log[x]^2 + Pi^2) Log[1 + x]^2 -
  2 / 3 (4 Log[x]^3 + 5 Pi^2 Log[x] - 6 Zeta[3]) Log[1 + x]
```

```
K[x_] := K0t[x] + K1t[x]
```

```
K1 = MBintegrate[Kmb, {x → 1 / 5}, Complex → True]
```

```
Shifting contours...
```

```
Performing 9 lower-dimensional integrations with NIntegrate...1...2...3...4...5...6...7...8
```

```
Higher-dimensional integrals
```

$$\left\{(296.731 + 0.\, i) - \frac{4.}{eps^4} - \frac{8.04719}{eps^3} + \frac{19.4934}{eps^2} + \frac{122.742 + 0.\, i}{eps},\ 0\right\}$$

```
K2 = K[1 / 5] // N
```

$$\left(296.731 + 4.06505 \times 10^{-15}\, i\right) - \frac{4.}{eps^4} - \frac{8.04719}{eps^3} + \frac{19.4934}{eps^2} + \frac{122.742}{eps}$$

```
K1[[1]] - K2
```

$$\left(-1.08571 \times 10^{-10} - 4.06505 \times 10^{-15}\, i\right) + \frac{8.52651 \times 10^{-14} + 0.\, i}{eps}$$

Derivation of sums at x = 1/15

```
<< MBsums.v1.0.m
```

```
MBsums v1.0 by Michal Ochman
```

```
The author would like to thank Tord Riemann
for many fruitful discussions
```

n::shdw : Symbol n appears in multiple contexts {MBsums`, Global`};
definitions in context MBsums` may shadow or be shadowed by other definitions. ≫

```
Lk = {x → 1 / 15};
```

```
kmax = Length[Kmb];
SResult = {};
Do[
egz = Kmb[[k]];
zL = egz[[2, 2]] /. (z_ → _) :→ (z → L);
s1 = MBIntToSum[egz, Lk, zL];
num1 = DoAllMBSums[s1, 130, Lk] // Expand // N;
num2 = MBintegrate[{egz}, Lk];
  SResult = Append[SResult, s1]; Print[num1, "\n", num2], {k, kmax}];

sums = Flatten[SResult, 1]
```

Shifting contours...

Performing 0 lower-dimensional integrations with NIntegratePerforming 1 lower-dimensional :

Higher-dimensional integrals

$563.844 - \frac{4.}{eps^4} - \frac{13.5403}{eps^3} + \frac{10.0069}{eps^2} + \frac{152.938}{eps}$

$\left\{563.844 - \frac{4.}{eps^4} - \frac{13.5403}{eps^3} + \frac{10.0069}{eps^2} + \frac{152.938}{eps}, 0\right\}$

z1->L (Re z1 < -15/128)

Shifting contours...

Higher-dimensional integrals

-34.0665
{-34.32251811524131, 0}

z2->L (Re z2 < -79/128)

Shifting contours...

Higher-dimensional integrals

-39.5482
{-39.80371049794612, 0}

z2->L (Re z2 < -1/2)

Shifting contours...

Higher-dimensional integrals

-5.90248
{-5.952777785529439, 0}

z3->L (Re z3 < -15/16)

Shifting contours...

Higher-dimensional integrals

35.7366
{35.73659230027591, 0}

z3->L (Re z3 < -7/16)

Shifting contours...

Higher-dimensional integrals

```
35.7366
{35.73659230027591, 0}
z3->L ( Re z3 < -41/128 )
Shifting contours...
Higher-dimensional integrals
```

$134.459 + \frac{34.945}{\text{eps}}$

$\left\{134.4588357402644 + \frac{34.94498146991600}{\text{eps}}, 0\right\}$

```
z4->L ( Re z4 < -131/128 )
Shifting contours...
Higher-dimensional integrals
```

$3.03718 + \frac{1.59774}{\text{eps}}$

$\left\{3.037181974942420 + \frac{1.597736171319681}{\text{eps}}, 0\right\}$

$$\Big\{\text{MBsum}\Big[\frac{1}{720\,\text{eps}^4}\big(-2880 + 1800\,\text{eps}^2\,\pi^2 + 180\,\text{eps}^4\,\text{EulerGamma}^2\,\pi^2 + 923\,\text{eps}^4\,\pi^4 + 3600\,\text{eps}\,\text{Log}[x] - 3240\,\text{eps}^3\,\pi^2\,\text{Log}[x] - 1440\,\text{eps}^2\,\text{Log}[x]^2 + 3000\,\text{eps}^4\,\pi^2\,\text{Log}[x]^2 - 240\,\text{eps}^3\,\text{Log}[x]^3 + 600\,\text{eps}^4\,\text{Log}[x]^4 + 15\,600\,\text{eps}^3\,\text{Zeta}[3] + 2880\,\text{eps}^4\,\text{EulerGamma}\,\text{Zeta}[3] - 19\,680\,\text{eps}^4\,\text{Log}[x]\,\text{Zeta}[3]\big),$$

$$\text{True}, \{\}\Big], \text{MBsum}\Big[\frac{1}{12\,(\text{n1}!)^2}\,(-1)^{-3\,\text{n1}}\,(-1+\text{n1})!$$

$$\big(4\,(-1)^{\text{n1}}\,\pi^2\,(-1+\text{n1})! + 24\,\pi^2\,x^{\text{n1}}\,(-1+\text{n1})! - 3\,(-1)^{\text{n1}}\,\text{EulerGamma}\,\pi^2\,\text{n1}! - 48\,(-1)^{\text{n1}}\,\text{EulerGamma}\,(-1+\text{n1})!\,\text{HarmonicNumber}[-1+\text{n1}] + 9\,(-1)^{\text{n1}}\,\text{EulerGamma}^2\,\text{n1}!\,\text{HarmonicNumber}[-1+\text{n1}] + 22\,(-1)^{\text{n1}}\,\pi^2\,\text{n1}!\,\text{HarmonicNumber}[-1+\text{n1}] + 48\,(-1)^{\text{n1}}\,(-1+\text{n1})!\,\text{HarmonicNumber}[-1+\text{n1}]^2 + 96\,x^{\text{n1}}\,(-1+\text{n1})!\,\text{HarmonicNumber}[-1+\text{n1}]^2 - 33\,(-1)^{\text{n1}}\,\text{EulerGamma}\,\text{n1}!\,\text{HarmonicNumber}[-1+\text{n1}]^2 + 31\,(-1)^{\text{n1}}\,\text{n1}!\,\text{HarmonicNumber}[-1+\text{n1}]^3 + 48\,(-1)^{\text{n1}}\,\text{EulerGamma}\,(-1+\text{n1})!\,\text{HarmonicNumber}[\text{n1}] - 9\,(-1)^{\text{n1}}\,\text{EulerGamma}^2\,\text{n1}!\,\text{HarmonicNumber}[\text{n1}] - 19\,(-1)^{\text{n1}}\,\pi^2\,\text{n1}!\,\text{HarmonicNumber}[\text{n1}] - 48\,(-1)^{\text{n1}}\,(-1+\text{n1})!\,\text{HarmonicNumber}[-1+\text{n1}]\,\text{HarmonicNumber}[\text{n1}] - 192\,x^{\text{n1}}\,(-1+\text{n1})!\,\text{HarmonicNumber}[-1+\text{n1}]\,\text{HarmonicNumber}[\text{n1}] + 48\,(-1)^{\text{n1}}\,\text{EulerGamma}\,\text{n1}!\,\text{HarmonicNumber}[-1+\text{n1}]\,\text{HarmonicNumber}[\text{n1}] - 60\,(-1)^{\text{n1}}\,\text{n1}!\,\text{HarmonicNumber}[-1+\text{n1}]^2\,\text{HarmonicNumber}[\text{n1}] + 96\,x^{\text{n1}}\,(-1+\text{n1})!\,\text{HarmonicNumber}[\text{n1}]^2 - 15\,(-1)^{\text{n1}}\,\text{EulerGamma}\,\text{n1}!\,\text{HarmonicNumber}[\text{n1}]^2 + 36\,(-1)^{\text{n1}}\,\text{n1}!\,\text{HarmonicNumber}[-1+\text{n1}]\,\text{HarmonicNumber}[\text{n1}]^2 - 7\,(-1)^{\text{n1}}\,\text{n1}!\,\text{HarmonicNumber}[\text{n1}]^3 - 24\,(-1)^{\text{n1}}\,(-1+\text{n1})!\,\text{HarmonicNumber}[-1+\text{n1}, 2] - 48\,x^{\text{n1}}\,(-1+\text{n1})!\,\text{HarmonicNumber}[-1+\text{n1}, 2] + 33\,(-1)^{\text{n1}}\,\text{EulerGamma}\,\text{n1}!\,\text{HarmonicNumber}[-1+\text{n1}, 2] - 78\,(-1)^{\text{n1}}\,\text{n1}!\,\text{HarmonicNumber}[-1+\text{n1}]\,\text{HarmonicNumber}[-1+\text{n1}, 2] + 45\,(-1)^{\text{n1}}\,\text{n1}!\,\text{HarmonicNumber}[\text{n1}]\,\text{HarmonicNumber}[-1+\text{n1}, 2] + 32\,(-1)^{\text{n1}}\,\text{n1}!\,\text{HarmonicNumber}[-1+\text{n1}, 3] + 48\,x^{\text{n1}}\,(-1+\text{n1})!\,\text{HarmonicNumber}[\text{n1}, 2] - 15\,(-1)^{\text{n1}}\,\text{EulerGamma}\,\text{n1}!\,\text{HarmonicNumber}[\text{n1}, 2] +$$

```
60 (-1)^n1 n1! HarmonicNumber[-1 + n1] HarmonicNumber[n1, 2] -
45 (-1)^n1 n1! HarmonicNumber[n1] HarmonicNumber[n1, 2] -
62 (-1)^n1 n1! HarmonicNumber[n1, 3] + 96 x^n1 (-1 + n1)!
 HarmonicNumber[-1 + n1] Log[x] - 96 x^n1 (-1 + n1)! HarmonicNumber[n1] Log[x] +
24 x^n1 (-1 + n1)! Log[x]^2 + 30 (-1)^n1 n1! Zeta[3]), n1 ≥ 1, {n1}],
MBsum[-1/(12 (n1!)^2) (-1)^(-3 n1) (-1 + n1)! (-24 π^2 x^n1 (-1 + n1)! + 3 (-1)^n1 EulerGamma π^2 n1! -
  9 (-1)^n1 EulerGamma^2 n1! HarmonicNumber[-1 + n1] -
  22 (-1)^n1 π^2 n1! HarmonicNumber[-1 + n1] - 96 x^n1 (-1 + n1)!
   HarmonicNumber[-1 + n1]^2 + 33 (-1)^n1 EulerGamma n1! HarmonicNumber[-1 + n1]^2 -
  31 (-1)^n1 n1! HarmonicNumber[-1 + n1]^3 + 9 (-1)^n1 EulerGamma^2 n1!
   HarmonicNumber[n1] + 19 (-1)^n1 π^2 n1! HarmonicNumber[n1] +
  192 x^n1 (-1 + n1)! HarmonicNumber[-1 + n1] HarmonicNumber[n1] -
  48 (-1)^n1 EulerGamma n1! HarmonicNumber[-1 + n1] HarmonicNumber[n1] +
  60 (-1)^n1 n1! HarmonicNumber[-1 + n1]^2 HarmonicNumber[n1] - 96 x^n1 (-1 + n1)!
   HarmonicNumber[n1]^2 + 15 (-1)^n1 EulerGamma n1! HarmonicNumber[n1]^2 -
  36 (-1)^n1 n1! HarmonicNumber[-1 + n1] HarmonicNumber[n1]^2 +
  7 (-1)^n1 n1! HarmonicNumber[n1]^3 + 48 x^n1 (-1 + n1)! HarmonicNumber[-1 + n1, 2] -
  33 (-1)^n1 EulerGamma n1! HarmonicNumber[-1 + n1, 2] +
  78 (-1)^n1 n1! HarmonicNumber[-1 + n1] HarmonicNumber[-1 + n1, 2] -
  45 (-1)^n1 n1! HarmonicNumber[n1] HarmonicNumber[-1 + n1, 2] -
  32 (-1)^n1 n1! HarmonicNumber[-1 + n1, 3] - 48 x^n1 (-1 + n1)!
   HarmonicNumber[n1, 2] + 15 (-1)^n1 EulerGamma n1! HarmonicNumber[n1, 2] -
  60 (-1)^n1 n1! HarmonicNumber[-1 + n1] HarmonicNumber[n1, 2] +
  45 (-1)^n1 n1! HarmonicNumber[n1] HarmonicNumber[n1, 2] +
  62 (-1)^n1 n1! HarmonicNumber[n1, 3] - 96 x^n1 (-1 + n1)! HarmonicNumber[-1 + n1]
   Log[x] + 96 x^n1 (-1 + n1)! HarmonicNumber[n1] Log[x] -
  24 x^n1 (-1 + n1)! Log[x]^2 - 30 (-1)^n1 n1! Zeta[3]), n1 ≥ 1, {n1}],
MBsum[1/(6 n1!) (-1)^(-2 n1) (-1 + n1)! (2 π^2 HarmonicNumber[-1 + n1] + HarmonicNumber[-1 + n1]^3 -
 2 π^2 HarmonicNumber[n1] - 3 HarmonicNumber[-1 + n1]^2 HarmonicNumber[n1] +
 3 HarmonicNumber[-1 + n1] HarmonicNumber[n1]^2 - HarmonicNumber[n1]^3 +
 3 HarmonicNumber[-1 + n1] HarmonicNumber[-1 + n1, 2] -
 3 HarmonicNumber[n1] HarmonicNumber[-1 + n1, 2] -
 10 HarmonicNumber[-1 + n1, 3] + 9 HarmonicNumber[-1 + n1] HarmonicNumber[n1, 2] -
 9 HarmonicNumber[n1] HarmonicNumber[n1, 2] -
 14 HarmonicNumber[n1, 3] + 24 Zeta[3]), n1 ≥ 1, {n1}],
MBsum[1/(3 n1!) (-1)^(-3 n1) x^n1 (-1 + n1)! (π^2 HarmonicNumber[-1 + n1] +
 2 π^2 HarmonicNumber[n1] + 3 HarmonicNumber[-1 + n1]^2 HarmonicNumber[n1] -
 6 HarmonicNumber[-1 + n1] HarmonicNumber[n1]^2 + 3 HarmonicNumber[n1]^3 -
 3 HarmonicNumber[n1] HarmonicNumber[-1 + n1, 2] -
 6 HarmonicNumber[-1 + n1] HarmonicNumber[n1, 2] +
 9 HarmonicNumber[n1] HarmonicNumber[n1, 2] + 6 HarmonicNumber[n1, 3] +
 π^2 Log[x] + 6 HarmonicNumber[-1 + n1] HarmonicNumber[n1] Log[x] -
 6 HarmonicNumber[n1]^2 Log[x] - 6 HarmonicNumber[n1, 2] Log[x] +
```

```
    3 HarmonicNumber[n1] Log[x]^2 - 6 Zeta[3]), n1 ≥ 1, {n1}],
MBsum[1/(90 (-n1)! n1!) x^-n1 (17 π^4 - 75 π^2 HarmonicNumber[-n1]^2 -
    30 HarmonicNumber[-n1]^4 + 60 π^2 HarmonicNumber[-n1] HarmonicNumber[n1] +
    90 HarmonicNumber[-n1]^3 HarmonicNumber[n1] + 15 π^2 HarmonicNumber[n1]^2 -
    90 HarmonicNumber[-n1]^2 HarmonicNumber[n1]^2 +
    30 HarmonicNumber[-n1] HarmonicNumber[n1]^3 - 75 π^2 HarmonicNumber[-n1, 2] -
    180 HarmonicNumber[-n1]^2 HarmonicNumber[-n1, 2] +
    270 HarmonicNumber[-n1] HarmonicNumber[n1] HarmonicNumber[-n1, 2] -
    90 HarmonicNumber[n1]^2 HarmonicNumber[-n1, 2] - 90 HarmonicNumber[-n1, 2]^2 -
    240 HarmonicNumber[-n1] HarmonicNumber[-n1, 3] +
    180 HarmonicNumber[n1] HarmonicNumber[-n1, 3] - 180 HarmonicNumber[-n1, 4] +
    15 π^2 HarmonicNumber[n1, 2] - 90 HarmonicNumber[-n1]^2 HarmonicNumber[n1, 2] +
    90 HarmonicNumber[-n1] HarmonicNumber[n1] HarmonicNumber[n1, 2] -
    90 HarmonicNumber[-n1, 2] HarmonicNumber[n1, 2] + 60 HarmonicNumber[-n1]
     HarmonicNumber[n1, 3] + 60 π^2 HarmonicNumber[-n1] Log[x] +
    90 HarmonicNumber[-n1]^3 Log[x] + 30 π^2 HarmonicNumber[n1] Log[x] -
    180 HarmonicNumber[-n1]^2 HarmonicNumber[n1] Log[x] +
    90 HarmonicNumber[-n1] HarmonicNumber[n1]^2 Log[x] +
    270 HarmonicNumber[-n1] HarmonicNumber[-n1, 2] Log[x] - 180 HarmonicNumber[n1]
     HarmonicNumber[-n1, 2] Log[x] + 180 HarmonicNumber[-n1, 3] Log[x] +
    90 HarmonicNumber[-n1] HarmonicNumber[n1, 2] Log[x] + 15 π^2 Log[x]^2 -
    90 HarmonicNumber[-n1]^2 Log[x]^2 + 90 HarmonicNumber[-n1]
     HarmonicNumber[n1] Log[x]^2 - 90 HarmonicNumber[-n1, 2] Log[x]^2 +
    30 HarmonicNumber[-n1] Log[x]^3 + 180 HarmonicNumber[-n1] Zeta[3] -
    180 HarmonicNumber[n1] Zeta[3] - 180 Log[x] Zeta[3]), n1 == 0, {n1}],
MBsum[1/(3 n1!) (-1)^(-3 n1) x^n1 (-1 + n1)! (π^2 HarmonicNumber[-1 + n1] +
    2 π^2 HarmonicNumber[n1] + 3 HarmonicNumber[-1 + n1]^2 HarmonicNumber[n1] -
    6 HarmonicNumber[-1 + n1] HarmonicNumber[n1]^2 + 3 HarmonicNumber[n1]^3 -
    3 HarmonicNumber[n1] HarmonicNumber[-1 + n1, 2] -
    6 HarmonicNumber[-1 + n1] HarmonicNumber[n1, 2] +
    9 HarmonicNumber[n1] HarmonicNumber[n1, 2] + 6 HarmonicNumber[n1, 3] +
    π^2 Log[x] + 6 HarmonicNumber[-1 + n1] HarmonicNumber[n1] Log[x] -
    6 HarmonicNumber[n1]^2 Log[x] - 6 HarmonicNumber[n1, 2] Log[x] +
    3 HarmonicNumber[n1] Log[x]^2 - 6 Zeta[3]), n1 ≥ 1, {n1}],
MBsum[1/(90 (-n1)! n1!) x^-n1 (17 π^4 - 75 π^2 HarmonicNumber[-n1]^2 -
    30 HarmonicNumber[-n1]^4 + 60 π^2 HarmonicNumber[-n1] HarmonicNumber[n1] +
    90 HarmonicNumber[-n1]^3 HarmonicNumber[n1] + 15 π^2 HarmonicNumber[n1]^2 -
    90 HarmonicNumber[-n1]^2 HarmonicNumber[n1]^2 +
    30 HarmonicNumber[-n1] HarmonicNumber[n1]^3 - 75 π^2 HarmonicNumber[-n1, 2] -
    180 HarmonicNumber[-n1]^2 HarmonicNumber[-n1, 2] +
    270 HarmonicNumber[-n1] HarmonicNumber[n1] HarmonicNumber[-n1, 2] -
    90 HarmonicNumber[n1]^2 HarmonicNumber[-n1, 2] - 90 HarmonicNumber[-n1, 2]^2 -
    240 HarmonicNumber[-n1] HarmonicNumber[-n1, 3] +
    180 HarmonicNumber[n1] HarmonicNumber[-n1, 3] - 180 HarmonicNumber[-n1, 4] +
```

```
15 π^2 HarmonicNumber[n1, 2] - 90 HarmonicNumber[-n1]^2 HarmonicNumber[n1, 2] +
90 HarmonicNumber[-n1] HarmonicNumber[n1] HarmonicNumber[n1, 2] -
90 HarmonicNumber[-n1, 2] HarmonicNumber[n1, 2] + 60 HarmonicNumber[-n1]
 HarmonicNumber[n1, 3] + 60 π^2 HarmonicNumber[-n1] Log[x] +
90 HarmonicNumber[-n1]^3 Log[x] + 30 π^2 HarmonicNumber[n1] Log[x] -
180 HarmonicNumber[-n1]^2 HarmonicNumber[n1] Log[x] +
90 HarmonicNumber[-n1] HarmonicNumber[n1]^2 Log[x] +
270 HarmonicNumber[-n1] HarmonicNumber[-n1, 2] Log[x] - 180 HarmonicNumber[n1]
 HarmonicNumber[-n1, 2] Log[x] + 180 HarmonicNumber[-n1, 3] Log[x] +
90 HarmonicNumber[-n1] HarmonicNumber[n1, 2] Log[x] + 15 π^2 Log[x]^2 -
90 HarmonicNumber[-n1]^2 Log[x]^2 + 90 HarmonicNumber[-n1]
 HarmonicNumber[n1] Log[x]^2 - 90 HarmonicNumber[-n1, 2] Log[x]^2 +
30 HarmonicNumber[-n1] Log[x]^3 + 180 HarmonicNumber[-n1] Zeta[3] -
180 HarmonicNumber[n1] Zeta[3] - 180 Log[x] Zeta[3]),
n1 == 0, {n1}], MBsum[- 1/(3 eps n1!) (-1)^(-3 n1) x^n1 (-1 + n1)!
 (3 π^2 + 3 eps π^2 HarmonicNumber[-1 + n1] + 3 HarmonicNumber[-1 + n1]^2 +
  2 eps HarmonicNumber[-1 + n1]^3 - 6 HarmonicNumber[-1 + n1] HarmonicNumber[n1] -
  3 eps HarmonicNumber[-1 + n1]^2 HarmonicNumber[n1] + 3 HarmonicNumber[n1]^2 +
  eps HarmonicNumber[n1]^3 - 3 HarmonicNumber[-1 + n1, 2] -
  6 eps HarmonicNumber[-1 + n1] HarmonicNumber[-1 + n1, 2] +
  3 eps HarmonicNumber[n1] HarmonicNumber[-1 + n1, 2] +
  4 eps HarmonicNumber[-1 + n1, 3] + 3 HarmonicNumber[n1, 2] +
  3 eps HarmonicNumber[n1] HarmonicNumber[n1, 2] + 2 eps HarmonicNumber[n1, 3] -
  3 eps π^2 Log[x] + 6 HarmonicNumber[-1 + n1] Log[x] - 6 HarmonicNumber[n1] Log[x] +
  6 eps HarmonicNumber[-1 + n1] HarmonicNumber[n1] Log[x] -
  6 eps HarmonicNumber[n1]^2 Log[x] - 6 eps HarmonicNumber[n1, 2] Log[x] +
  3 Log[x]^2 - 6 eps HarmonicNumber[-1 + n1] Log[x]^2 +
  9 eps HarmonicNumber[n1] Log[x]^2 - 4 eps Log[x]^3 - 6 eps Zeta[3]), n1 ≥ 1, {n1}],
MBsum[1/(6 eps (-n1)! n1!) x^(-n1) (6 π^2 HarmonicNumber[-n1] + 3 eps π^2 HarmonicNumber[-n1]^2 +
 2 HarmonicNumber[-n1]^3 + eps HarmonicNumber[-n1]^4 -
 6 π^2 HarmonicNumber[n1] - 6 HarmonicNumber[-n1]^2 HarmonicNumber[n1] -
 2 eps HarmonicNumber[-n1]^3 HarmonicNumber[n1] - 3 eps π^2 HarmonicNumber[n1]^2 +
 6 HarmonicNumber[-n1] HarmonicNumber[n1]^2 - 2 HarmonicNumber[n1]^3 +
 2 eps HarmonicNumber[-n1] HarmonicNumber[n1]^3 - eps HarmonicNumber[n1]^4 +
 3 eps π^2 HarmonicNumber[-n1, 2] + 6 HarmonicNumber[-n1] HarmonicNumber[-n1, 2] +
 6 eps HarmonicNumber[-n1]^2 HarmonicNumber[-n1, 2] -
 6 HarmonicNumber[n1] HarmonicNumber[-n1, 2] -
 6 eps HarmonicNumber[-n1] HarmonicNumber[n1] HarmonicNumber[-n1, 2] +
 3 eps HarmonicNumber[-n1, 2]^2 + 4 HarmonicNumber[-n1, 3] +
 8 eps HarmonicNumber[-n1] HarmonicNumber[-n1, 3] -
 4 eps HarmonicNumber[n1] HarmonicNumber[-n1, 3] + 6 eps HarmonicNumber[-n1, 4] -
 3 eps π^2 HarmonicNumber[n1, 2] + 6 HarmonicNumber[-n1] HarmonicNumber[n1, 2] -
 6 HarmonicNumber[n1] HarmonicNumber[n1, 2] +
 6 eps HarmonicNumber[-n1] HarmonicNumber[n1] HarmonicNumber[n1, 2] -
 6 eps HarmonicNumber[n1]^2 HarmonicNumber[n1, 2] - 3 eps HarmonicNumber[n1, 2]^2 -
```

```
    4 HarmonicNumber[n1, 3] + 4 eps HarmonicNumber[-n1] HarmonicNumber[n1, 3] -
    8 eps HarmonicNumber[n1] HarmonicNumber[n1, 3] -
    6 eps HarmonicNumber[n1, 4] - 6 π^2 Log[x] -
    12 eps π^2 HarmonicNumber[-n1] Log[x] - 6 HarmonicNumber[-n1]^2 Log[x] -
    6 eps HarmonicNumber[-n1]^3 Log[x] + 6 eps π^2 HarmonicNumber[n1] Log[x] +
    12 HarmonicNumber[-n1] HarmonicNumber[n1] Log[x] +
    12 eps HarmonicNumber[-n1]^2 HarmonicNumber[n1] Log[x] -
    6 HarmonicNumber[n1]^2 Log[x] - 6 eps HarmonicNumber[-n1]
     HarmonicNumber[n1]^2 Log[x] - 6 HarmonicNumber[-n1, 2] Log[x] -
    18 eps HarmonicNumber[-n1] HarmonicNumber[-n1, 2] Log[x] +
    12 eps HarmonicNumber[n1] HarmonicNumber[-n1, 2] Log[x] -
    12 eps HarmonicNumber[-n1, 3] Log[x] - 6 HarmonicNumber[n1, 2] Log[x] -
    6 eps HarmonicNumber[-n1] HarmonicNumber[n1, 2] Log[x] +
    9 eps π^2 Log[x]^2 + 6 HarmonicNumber[-n1] Log[x]^2 +
    12 eps HarmonicNumber[-n1]^2 Log[x]^2 - 6 HarmonicNumber[n1] Log[x]^2 -
    18 eps HarmonicNumber[-n1] HarmonicNumber[n1] Log[x]^2 +
    6 eps HarmonicNumber[n1]^2 Log[x]^2 + 12 eps HarmonicNumber[-n1, 2] Log[x]^2 +
    6 eps HarmonicNumber[n1, 2] Log[x]^2 - 2 Log[x]^3 -
    10 eps HarmonicNumber[-n1] Log[x]^3 + 8 eps HarmonicNumber[n1] Log[x]^3 +
    3 eps Log[x]^4 - 12 eps HarmonicNumber[-n1] Zeta[3] +
    12 eps HarmonicNumber[n1] Zeta[3] + 12 eps Log[x] Zeta[3]), n1 == 0, {n1}],
MBsum[- 1/(3 eps n1!) (-1)^(-3 n1) x^n1 (-1 + n1)! (3 π^2 + eps π^2 HarmonicNumber[-1 + n1] +
     3 HarmonicNumber[-1 + n1]^2 + 2 eps HarmonicNumber[-1 + n1]^3 -
     4 eps π^2 HarmonicNumber[n1] - 6 HarmonicNumber[-1 + n1] HarmonicNumber[n1] -
     9 eps HarmonicNumber[-1 + n1]^2 HarmonicNumber[n1] + 3 HarmonicNumber[n1]^2 +
     12 eps HarmonicNumber[-1 + n1] HarmonicNumber[n1]^2 -
     5 eps HarmonicNumber[n1]^3 - 3 HarmonicNumber[-1 + n1, 2] -
     6 eps HarmonicNumber[-1 + n1] HarmonicNumber[-1 + n1, 2] +
     9 eps HarmonicNumber[n1] HarmonicNumber[-1 + n1, 2] +
     4 eps HarmonicNumber[-1 + n1, 3] + 3 HarmonicNumber[n1, 2] +
     12 eps HarmonicNumber[-1 + n1] HarmonicNumber[n1, 2] -
     15 eps HarmonicNumber[n1] HarmonicNumber[n1, 2] - 10 eps HarmonicNumber[n1, 3] -
     5 eps π^2 Log[x] + 6 HarmonicNumber[-1 + n1] Log[x] - 6 HarmonicNumber[n1] Log[x] -
     6 eps HarmonicNumber[-1 + n1] HarmonicNumber[n1] Log[x] +
     6 eps HarmonicNumber[n1]^2 Log[x] + 6 eps HarmonicNumber[n1, 2] Log[x] +
     3 Log[x]^2 - 6 eps HarmonicNumber[-1 + n1] Log[x]^2 +
     3 eps HarmonicNumber[n1] Log[x]^2 - 4 eps Log[x]^3 + 6 eps Zeta[3]), n1 ≥ 1, {n1}]}
```

```
(* Simplify all sums *)

s2 = SimplifyMBsums[sums]
```

```
{MBsum[
  1/(6 eps (k1!)^2) (-1)^(-3 k1) (-1 + k1)! (2 (-1)^k1 eps π^2 (-1 + k1)! + 24 eps π^2 x^k1 (-1 + k1)! -
```

```
3 (-1)^k1 eps EulerGamma π^2 k1 ! - 12 π^2 x^k1 k1 ! - 24 (-1)^k1 eps EulerGamma (-1 + k1) !
 HarmonicNumber[-1 + k1] + 9 (-1)^k1 eps EulerGamma^2 k1 ! HarmonicNumber[-1 + k1] +
24 (-1)^k1 eps π^2 k1 ! HarmonicNumber[-1 + k1] - 4 eps π^2 x^k1 k1 !
 HarmonicNumber[-1 + k1] + 24 (-1)^k1 eps (-1 + k1) ! HarmonicNumber[-1 + k1]^2 +
96 eps x^k1 (-1 + k1) ! HarmonicNumber[-1 + k1]^2 - 33 (-1)^k1 eps EulerGamma
 k1 ! HarmonicNumber[-1 + k1]^2 - 12 x^k1 k1 ! HarmonicNumber[-1 + k1]^2 +
32 (-1)^k1 eps k1 ! HarmonicNumber[-1 + k1]^3 - 8 eps x^k1 k1 ! HarmonicNumber[-1 + k1]^3 +
24 (-1)^k1 eps EulerGamma (-1 + k1) ! HarmonicNumber[k1] -
9 (-1)^k1 eps EulerGamma^2 k1 ! HarmonicNumber[k1] -
21 (-1)^k1 eps π^2 k1 ! HarmonicNumber[k1] + 16 eps π^2 x^k1 k1 ! HarmonicNumber[k1] -
24 (-1)^k1 eps (-1 + k1) ! HarmonicNumber[-1 + k1] HarmonicNumber[k1] -
192 eps x^k1 (-1 + k1) ! HarmonicNumber[-1 + k1] HarmonicNumber[k1] +
48 (-1)^k1 eps EulerGamma k1 ! HarmonicNumber[-1 + k1] HarmonicNumber[k1] +
24 x^k1 k1 ! HarmonicNumber[-1 + k1] HarmonicNumber[k1] -
63 (-1)^k1 eps k1 ! HarmonicNumber[-1 + k1]^2 HarmonicNumber[k1] +
36 eps x^k1 k1 ! HarmonicNumber[-1 + k1]^2 HarmonicNumber[k1] +
96 eps x^k1 (-1 + k1) ! HarmonicNumber[k1]^2 -
15 (-1)^k1 eps EulerGamma k1 ! HarmonicNumber[k1]^2 - 12 x^k1 k1 ! HarmonicNumber[k1]^2 +
39 (-1)^k1 eps k1 ! HarmonicNumber[-1 + k1] HarmonicNumber[k1]^2 -
48 eps x^k1 k1 ! HarmonicNumber[-1 + k1] HarmonicNumber[k1]^2 -
8 (-1)^k1 eps k1 ! HarmonicNumber[k1]^3 + 20 eps x^k1 k1 ! HarmonicNumber[k1]^3 -
12 (-1)^k1 eps (-1 + k1) ! HarmonicNumber[-1 + k1, 2] -
48 eps x^k1 (-1 + k1) ! HarmonicNumber[-1 + k1, 2] + 33 (-1)^k1 eps EulerGamma
 k1 ! HarmonicNumber[-1 + k1, 2] + 12 x^k1 k1 ! HarmonicNumber[-1 + k1, 2] -
75 (-1)^k1 eps k1 ! HarmonicNumber[-1 + k1] HarmonicNumber[-1 + k1, 2] +
24 eps x^k1 k1 ! HarmonicNumber[-1 + k1] HarmonicNumber[-1 + k1, 2] +
42 (-1)^k1 eps k1 ! HarmonicNumber[k1] HarmonicNumber[-1 + k1, 2] -
36 eps x^k1 k1 ! HarmonicNumber[k1] HarmonicNumber[-1 + k1, 2] + 22 (-1)^k1 eps
 k1 ! HarmonicNumber[-1 + k1, 3] - 16 eps x^k1 k1 ! HarmonicNumber[-1 + k1, 3] +
48 eps x^k1 (-1 + k1) ! HarmonicNumber[k1, 2] - 15 (-1)^k1 eps EulerGamma
 k1 ! HarmonicNumber[k1, 2] - 12 x^k1 k1 ! HarmonicNumber[k1, 2] +
69 (-1)^k1 eps k1 ! HarmonicNumber[-1 + k1] HarmonicNumber[k1, 2] -
48 eps x^k1 k1 ! HarmonicNumber[-1 + k1] HarmonicNumber[k1, 2] -
54 (-1)^k1 eps k1 ! HarmonicNumber[k1] HarmonicNumber[k1, 2] +
60 eps x^k1 k1 ! HarmonicNumber[k1] HarmonicNumber[k1, 2] -
76 (-1)^k1 eps k1 ! HarmonicNumber[k1, 3] + 40 eps x^k1 k1 ! HarmonicNumber[k1, 3] +
20 eps π^2 x^k1 k1 ! Log[x] + 96 eps x^k1 (-1 + k1) ! HarmonicNumber[-1 + k1] Log[x] -
24 x^k1 k1 ! HarmonicNumber[-1 + k1] Log[x] - 96 eps x^k1 (-1 + k1) !
 HarmonicNumber[k1] Log[x] + 24 x^k1 k1 ! HarmonicNumber[k1] Log[x] +
24 eps x^k1 k1 ! HarmonicNumber[-1 + k1] HarmonicNumber[k1] Log[x] -
24 eps x^k1 k1 ! HarmonicNumber[k1]^2 Log[x] -
24 eps x^k1 k1 ! HarmonicNumber[k1, 2] Log[x] + 24 eps x^k1 (-1 + k1) ! Log[x]^2 -
12 x^k1 k1 ! Log[x]^2 + 24 eps x^k1 k1 ! HarmonicNumber[-1 + k1] Log[x]^2 -
12 eps x^k1 k1 ! HarmonicNumber[k1] Log[x]^2 + 16 eps x^k1 k1 ! Log[x]^3 +
54 (-1)^k1 eps k1 ! Zeta[3] - 24 eps x^k1 k1 ! Zeta[3]), k1 ≥ 1, {k1}],
MBsum[1/(144 eps^4) (-576 + 360 eps^2 π^2 + 36 eps^4 EulerGamma^2 π^2 + 239 eps^4 π^4 +
```

$$720\,\text{eps}\,\text{Log}[x] - 792\,\text{eps}^3\,\pi^2\,\text{Log}[x] - 288\,\text{eps}^2\,\text{Log}[x]^2 + 864\,\text{eps}^4\,\pi^2\,\text{Log}[x]^2 - 96\,\text{eps}^3\,\text{Log}[x]^3 + 192\,\text{eps}^4\,\text{Log}[x]^4 + 3120\,\text{eps}^3\,\text{Zeta}[3] + 576\,\text{eps}^4\,\text{EulerGamma}\,\text{Zeta}[3] - 4224\,\text{eps}^4\,\text{Log}[x]\,\text{Zeta}[3]\Big),\ \text{True},\ \{\}\Big]\Big\}$$

```
(* Numerical check *)
```

```
DoAllMBSums[sums // ExpandAll, 2000, Lk] // Expand // N
```

$$692.796 - \frac{4.}{\text{eps}^4} - \frac{13.5403}{\text{eps}^3} + \frac{10.0069}{\text{eps}^2} + \frac{189.48}{\text{eps}}$$

```
DoAllMBSums[s2 // ExpandAll, 2000, Lk] // Expand // N
```

$$692.796 - \frac{4.}{\text{eps}^4} - \frac{13.5403}{\text{eps}^3} + \frac{10.0069}{\text{eps}^2} + \frac{189.48}{\text{eps}}$$

```
MBintegrate[Kmb, Lk, Complex → True]
```

```
Shifting contours...

Performing 9 lower-dimensional integrations with NIntegrate...1...2...3...4...5...6...7...8

Higher-dimensional integrals
```

$$\left\{(692.734 + 0.\,i) - \frac{4.}{\text{eps}^4} - \frac{13.5403}{\text{eps}^3} + \frac{10.0069}{\text{eps}^2} + \frac{189.48 + 0.\,i}{\text{eps}},\ 0\right\}$$

Towards analytic result

```
w = s2[[1]] /. k1 → k1 + 1 // Factor
```

$$\text{MBsum}\Big[\frac{1}{6\,\text{eps}\,((1+k1)!)^2}(-1)^{-3(1+k1)}\,k1!\,\Big(2(-1)^{1+k1}\,\text{eps}\,\pi^2\,k1! + 24\,\text{eps}\,\pi^2\,x^{1+k1}\,k1! - 3(-1)^{1+k1}\,\text{eps}\,\text{EulerGamma}\,\pi^2\,(1+k1)! - 12\,\pi^2\,x^{1+k1}\,(1+k1)! - 24(-1)^{1+k1}\,\text{eps}\,\text{EulerGamma}\,k1!\,\text{HarmonicNumber}[k1] + 9(-1)^{1+k1}\,\text{eps}\,\text{EulerGamma}^2\,(1+k1)!\,\text{HarmonicNumber}[k1] + 24(-1)^{1+k1}\,\text{eps}\,\pi^2\,(1+k1)!\,\text{HarmonicNumber}[k1] - 4\,\text{eps}\,\pi^2\,x^{1+k1}\,(1+k1)!\,\text{HarmonicNumber}[k1] + 24(-1)^{1+k1}\,\text{eps}\,k1!\,\text{HarmonicNumber}[k1]^2 + 96\,\text{eps}\,x^{1+k1}\,k1!\,\text{HarmonicNumber}[k1]^2 - 33(-1)^{1+k1}\,\text{eps}\,\text{EulerGamma}\,(1+k1)!\,\text{HarmonicNumber}[k1]^2 - 12\,x^{1+k1}\,(1+k1)!\,\text{HarmonicNumber}[k1]^2 + 32(-1)^{1+k1}\,\text{eps}\,(1+k1)!\,\text{HarmonicNumber}[k1]^3 - 8\,\text{eps}\,x^{1+k1}\,(1+k1)!\,\text{HarmonicNumber}[k1]^3 + 24(-1)^{1+k1}\,\text{eps}\,\text{EulerGamma}\,k1!\,\text{HarmonicNumber}[1+k1] - 9(-1)^{1+k1}\,\text{eps}\,\text{EulerGamma}^2\,(1+k1)!\,\text{HarmonicNumber}[1+k1] - 21(-1)^{1+k1}\,\text{eps}\,\pi^2\,(1+k1)!\,\text{HarmonicNumber}[1+k1] + 16\,\text{eps}\,\pi^2\,x^{1+k1}\,(1+k1)!\,\text{HarmonicNumber}[1+k1] - 24(-1)^{1+k1}\,\text{eps}\,k1!\,\text{HarmonicNumber}[k1]\,\text{HarmonicNumber}[1+k1] - 192\,\text{eps}\,x^{1+k1}\,k1!\,\text{HarmonicNumber}[k1]\,\text{HarmonicNumber}[1+k1] + 48(-1)^{1+k1}\,\text{eps}\,\text{EulerGamma}\,(1+k1)!\,\text{HarmonicNumber}[k1]\,\text{HarmonicNumber}[1+k1] + 24\,x^{1+k1}\,(1+k1)!\,\text{HarmonicNumber}[k1]\,\text{HarmonicNumber}[1+k1] - 63(-1)^{1+k1}\,\text{eps}\,(1+k1)!\,\text{HarmonicNumber}[k1]^2\,\text{HarmonicNumber}[1+k1] +$$

```
36 eps x^(1+k1) (1 + k1) ! HarmonicNumber[k1]^2 HarmonicNumber[1 + k1] +
96 eps x^(1+k1) k1 ! HarmonicNumber[1 + k1]^2 - 15 (-1)^(1+k1) eps EulerGamma (1 + k1) !
 HarmonicNumber[1 + k1]^2 - 12 x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1]^2 +
39 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[1 + k1]^2 -
48 eps x^(1+k1) (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[1 + k1]^2 -
8 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[1 + k1]^3 +
20 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1]^3 -
12 (-1)^(1+k1) eps k1 ! HarmonicNumber[k1, 2] - 48 eps x^(1+k1) k1 ! HarmonicNumber[k1, 2] +
33 (-1)^(1+k1) eps EulerGamma (1 + k1) ! HarmonicNumber[k1, 2] +
12 x^(1+k1) (1 + k1) ! HarmonicNumber[k1, 2] -
75 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[k1, 2] +
24 eps x^(1+k1) (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[k1, 2] +
42 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[1 + k1] HarmonicNumber[k1, 2] -
36 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1] HarmonicNumber[k1, 2] +
22 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[k1, 3] - 16 eps x^(1+k1) (1 + k1) !
 HarmonicNumber[k1, 3] + 48 eps x^(1+k1) k1 ! HarmonicNumber[1 + k1, 2] -
15 (-1)^(1+k1) eps EulerGamma (1 + k1) ! HarmonicNumber[1 + k1, 2] -
12 x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1, 2] +
69 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[1 + k1, 2] -
48 eps x^(1+k1) (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[1 + k1, 2] -
54 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[1 + k1] HarmonicNumber[1 + k1, 2] +
60 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1] HarmonicNumber[1 + k1, 2] -
76 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[1 + k1, 3] +
40 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1, 3] +
20 eps π^2 x^(1+k1) (1 + k1) ! Log[x] + 96 eps x^(1+k1) k1 ! HarmonicNumber[k1] Log[x] -
24 x^(1+k1) (1 + k1) ! HarmonicNumber[k1] Log[x] - 96 eps x^(1+k1) k1 !
 HarmonicNumber[1 + k1] Log[x] + 24 x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1] Log[x] +
24 eps x^(1+k1) (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[1 + k1] Log[x] -
24 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1]^2 Log[x] -
24 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1, 2] Log[x] + 24 eps x^(1+k1) k1 ! Log[x]^2 -
12 x^(1+k1) (1 + k1) ! Log[x]^2 + 24 eps x^(1+k1) (1 + k1) ! HarmonicNumber[k1] Log[x]^2 -
12 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1] Log[x]^2 + 16 eps x^(1+k1) (1 + k1) ! Log[x]^3 +
54 (-1)^(1+k1) eps (1 + k1) ! Zeta[3] - 24 eps x^(1+k1) (1 + k1) ! Zeta[3]), 1 + k1 ≥ 1, {1 + k1}]
```

w[[1]]

```
      1
---------------- (-1)^(-3 (1+k1)) k1 !
6 eps ((1 + k1) !)^2
 (2 (-1)^(1+k1) eps π^2 k1 ! + 24 eps π^2 x^(1+k1) k1 ! - 3 (-1)^(1+k1) eps EulerGamma π^2 (1 + k1) ! -
   12 π^2 x^(1+k1) (1 + k1) ! - 24 (-1)^(1+k1) eps EulerGamma k1 ! HarmonicNumber[k1] +
   9 (-1)^(1+k1) eps EulerGamma^2 (1 + k1) ! HarmonicNumber[k1] + 24 (-1)^(1+k1) eps π^2
    (1 + k1) ! HarmonicNumber[k1] - 4 eps π^2 x^(1+k1) (1 + k1) ! HarmonicNumber[k1] +
   24 (-1)^(1+k1) eps k1 ! HarmonicNumber[k1]^2 + 96 eps x^(1+k1) k1 ! HarmonicNumber[k1]^2 -
   33 (-1)^(1+k1) eps EulerGamma (1 + k1) ! HarmonicNumber[k1]^2 -
   12 x^(1+k1) (1 + k1) ! HarmonicNumber[k1]^2 + 32 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[k1]^3 -
   8 eps x^(1+k1) (1 + k1) ! HarmonicNumber[k1]^3 +
   24 (-1)^(1+k1) eps EulerGamma k1 ! HarmonicNumber[1 + k1] -
```

```
9 (-1)^(1+k1) eps EulerGamma^2 (1 + k1) ! HarmonicNumber[1 + k1] -
21 (-1)^(1+k1) eps π^2 (1 + k1) ! HarmonicNumber[1 + k1] +
16 eps π^2 x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1] -
24 (-1)^(1+k1) eps k1 ! HarmonicNumber[k1] HarmonicNumber[1 + k1] -
192 eps x^(1+k1) k1 ! HarmonicNumber[k1] HarmonicNumber[1 + k1] +
48 (-1)^(1+k1) eps EulerGamma (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[1 + k1] +
24 x^(1+k1) (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[1 + k1] -
63 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[k1]^2 HarmonicNumber[1 + k1] +
36 eps x^(1+k1) (1 + k1) ! HarmonicNumber[k1]^2 HarmonicNumber[1 + k1] +
96 eps x^(1+k1) k1 ! HarmonicNumber[1 + k1]^2 - 15 (-1)^(1+k1) eps EulerGamma (1 + k1) !
 HarmonicNumber[1 + k1]^2 - 12 x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1]^2 +
39 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[1 + k1]^2 -
48 eps x^(1+k1) (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[1 + k1]^2 -
8 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[1 + k1]^3 +
20 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1]^3 -
12 (-1)^(1+k1) eps k1 ! HarmonicNumber[k1, 2] - 48 eps x^(1+k1) k1 ! HarmonicNumber[k1, 2] +
33 (-1)^(1+k1) eps EulerGamma (1 + k1) ! HarmonicNumber[k1, 2] +
12 x^(1+k1) (1 + k1) ! HarmonicNumber[k1, 2] -
75 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[k1, 2] +
24 eps x^(1+k1) (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[k1, 2] +
42 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[1 + k1] HarmonicNumber[k1, 2] -
36 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1] HarmonicNumber[k1, 2] +
22 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[k1, 3] - 16 eps x^(1+k1) (1 + k1) !
 HarmonicNumber[k1, 3] + 48 eps x^(1+k1) k1 ! HarmonicNumber[1 + k1, 2] -
15 (-1)^(1+k1) eps EulerGamma (1 + k1) ! HarmonicNumber[1 + k1, 2] -
12 x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1, 2] +
69 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[1 + k1, 2] -
48 eps x^(1+k1) (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[1 + k1, 2] -
54 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[1 + k1] HarmonicNumber[1 + k1, 2] +
60 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1] HarmonicNumber[1 + k1, 2] -
76 (-1)^(1+k1) eps (1 + k1) ! HarmonicNumber[1 + k1, 3] +
40 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1, 3] +
20 eps π^2 x^(1+k1) (1 + k1) ! Log[x] + 96 eps x^(1+k1) k1 ! HarmonicNumber[k1] Log[x] -
24 x^(1+k1) (1 + k1) ! HarmonicNumber[k1] Log[x] - 96 eps x^(1+k1) k1 !
 HarmonicNumber[1 + k1] Log[x] + 24 x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1] Log[x] +
24 eps x^(1+k1) (1 + k1) ! HarmonicNumber[k1] HarmonicNumber[1 + k1] Log[x] -
24 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1]^2 Log[x] -
24 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1, 2] Log[x] + 24 eps x^(1+k1) k1 ! Log[x]^2 -
12 x^(1+k1) (1 + k1) ! Log[x]^2 + 24 eps x^(1+k1) (1 + k1) ! HarmonicNumber[k1] Log[x]^2 -
12 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1] Log[x]^2 + 16 eps x^(1+k1) (1 + k1) ! Log[x]^3 +
54 (-1)^(1+k1) eps (1 + k1) ! Zeta[3] - 24 eps x^(1+k1) (1 + k1) ! Zeta[3])
```

w1 = Coefficient[w[[1]], 1 / eps] // FullSimplify

$$\frac{1}{(1+k1)^3} 2\,(-1)^{-3\,k1}\, x^{1+k1} \left(2 + (1+k1)^2 \pi^2 + (1+k1)\,\mathrm{Log}[x]\,(-2 + (1+k1)\,\mathrm{Log}[x])\right)$$

```
w2 = w[[1]] - w1 / eps // Simplify

1/(6 eps)
 (-1)^(-3 k1) (-(12 x^(1+k1) (2 + (1 + k1)^2 π^2 + (1 + k1) Log[x] (-2 + (1 + k1) Log[x]))) / (1 + k1)^3 +
    1 / ((1 + k1) !)^2
     k1 ! (2 (-1)^k1 eps π^2 k1 ! - 24 eps π^2 x^(1+k1) k1 ! - 3 (-1)^k1 eps EulerGamma π^2 (1 + k1) ! +
        12 π^2 x^(1+k1) (1 + k1) ! + 8 eps (4 (-1)^k1 + x^(1+k1)) (1 + k1) ! HarmonicNumber[k1]^3 -
        4 eps (2 (-1)^k1 + 5 x^(1+k1)) (1 + k1) ! HarmonicNumber[1 + k1]^3 + 3 HarmonicNumber[k1]^2
         (8 eps ((-1)^k1 - 4 x^(1+k1)) k1 ! - (11 (-1)^k1 eps EulerGamma - 4 x^(1+k1)) (1 + k1) ! -
           3 eps (7 (-1)^k1 + 4 x^(1+k1)) (1 + k1) ! HarmonicNumber[1 + k1]) -
        12 (-1)^k1 eps k1 ! HarmonicNumber[k1, 2] + 48 eps x^(1+k1) k1 ! HarmonicNumber[k1, 2] +
        33 (-1)^k1 eps EulerGamma (1 + k1) ! HarmonicNumber[k1, 2] -
        12 x^(1+k1) (1 + k1) ! HarmonicNumber[k1, 2] + 22 (-1)^k1 eps (1 + k1) !
         HarmonicNumber[k1, 3] + 16 eps x^(1+k1) (1 + k1) ! HarmonicNumber[k1, 3] -
        48 eps x^(1+k1) k1 ! HarmonicNumber[1 + k1, 2] - 15 (-1)^k1 eps EulerGamma (1 + k1) !
         HarmonicNumber[1 + k1, 2] + 12 x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1, 2] -
        76 (-1)^k1 eps (1 + k1) ! HarmonicNumber[1 + k1, 3] -
        40 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1, 3] - 20 eps π^2 x^(1+k1) (1 + k1) ! Log[x] +
        24 eps x^(1+k1) (1 + k1) ! HarmonicNumber[1 + k1, 2] Log[x] - 24 eps x^(1+k1) k1 ! Log[x]^2 +
        12 x^(1+k1) (1 + k1) ! Log[x]^2 - 16 eps x^(1+k1) (1 + k1) ! Log[x]^3 + HarmonicNumber[1 + k1]
         (24 (-1)^k1 eps EulerGamma k1 ! - 9 (-1)^k1 eps EulerGamma^2 (1 + k1) ! -
           21 (-1)^k1 eps π^2 (1 + k1) ! - 16 eps π^2 x^(1+k1) (1 + k1) ! + 6 eps (7 (-1)^k1 + 6 x^(1+k1))
            (1 + k1) ! HarmonicNumber[k1, 2] - 6 eps (9 (-1)^k1 + 10 x^(1+k1)) (1 + k1) !
            HarmonicNumber[1 + k1, 2] + 96 eps x^(1+k1) k1 ! Log[x] - 24 x^(1+k1) (1 + k1) ! Log[x] +
           12 eps x^(1+k1) (1 + k1) ! Log[x]^2) - 3 HarmonicNumber[1 + k1]^2 (32 eps x^(1+k1) k1 ! +
           (1 + k1) ! (5 (-1)^k1 eps EulerGamma - 4 x^(1+k1) - 8 eps x^(1+k1) Log[x])) +
        HarmonicNumber[k1] (-24 (-1)^k1 eps EulerGamma k1 ! + 9 (-1)^k1 eps
            EulerGamma^2 (1 + k1) ! + 24 (-1)^k1 eps π^2 (1 + k1) ! + 4 eps π^2 x^(1+k1) (1 + k1) ! +
           3 eps (13 (-1)^k1 + 16 x^(1+k1)) (1 + k1) ! HarmonicNumber[1 + k1]^2 -
           3 eps (25 (-1)^k1 + 8 x^(1+k1)) (1 + k1) ! HarmonicNumber[k1, 2] +
           69 (-1)^k1 eps (1 + k1) ! HarmonicNumber[1 + k1, 2] + 48 eps x^(1+k1) (1 + k1) !
            HarmonicNumber[1 + k1, 2] - 96 eps x^(1+k1) k1 ! Log[x] + 24 x^(1+k1) (1 + k1) ! Log[x] -
           24 eps x^(1+k1) (1 + k1) ! Log[x]^2 - 24 HarmonicNumber[1 + k1] (eps ((-1)^k1 - 8 x^(1+k1))
               k1 ! - (1 + k1) ! (2 (-1)^k1 eps EulerGamma - x^(1+k1) - eps x^(1+k1) Log[x]))) +
        54 (-1)^k1 eps (1 + k1) ! Zeta[3] + 24 eps x^(1+k1) (1 + k1) ! Zeta[3]))

sw1 = Sum[w1, {k1, 0, Infinity}]

1/4 (-π^2 x^2 HypergeometricPFQ[{2, 2, 2, 2, 2}, {1, 3, 3, 3}, -x] -
   x^2 HypergeometricPFQ[{2, 2, 2, 2, 2}, {1, 3, 3, 3}, -x] Log[x]^2 -
   16 π^2 PolyLog[2, -x] + 16 Log[x] PolyLog[2, -x] - 16 Log[x]^2 PolyLog[2, -x] -
   16 PolyLog[3, -x] + 8 π^2 PolyLog[3, -x] + 8 Log[x]^2 PolyLog[3, -x])
```

sw1 = sw1 // FullSimplify

$2\left(\left(\pi^2+\text{Log}[x]^2\right)\text{Log}[1+x]+2\,\text{Log}[x]\,\text{PolyLog}[2,-x]-2\,\text{PolyLog}[3,-x]\right)$

s2[[2]]

$$\text{MBsum}\Big[\frac{1}{144\,\text{eps}^4}\left(-576+360\,\text{eps}^2\pi^2+36\,\text{eps}^4\,\text{EulerGamma}^2\pi^2+239\,\text{eps}^4\pi^4+720\,\text{eps}\,\text{Log}[x]-792\,\text{eps}^3\pi^2\,\text{Log}[x]-288\,\text{eps}^2\,\text{Log}[x]^2+864\,\text{eps}^4\pi^2\,\text{Log}[x]^2-96\,\text{eps}^3\,\text{Log}[x]^3+192\,\text{eps}^4\,\text{Log}[x]^4+3120\,\text{eps}^3\,\text{Zeta}[3]+576\,\text{eps}^4\,\text{EulerGamma}\,\text{Zeta}[3]-4224\,\text{eps}^4\,\text{Log}[x]\,\text{Zeta}[3]\right),\ \text{True},\ \{\}\Big]$$

sumresult = s2[[2, 1]] + sw1 / eps (* + Sum[w2,{k1,0,Infinity}] *)

$$\frac{1}{\text{eps}}2\left(\left(\pi^2+\text{Log}[x]^2\right)\text{Log}[1+x]+2\,\text{Log}[x]\,\text{PolyLog}[2,-x]-2\,\text{PolyLog}[3,-x]\right)+\frac{1}{144\,\text{eps}^4}\left(-576+360\,\text{eps}^2\pi^2+36\,\text{eps}^4\,\text{EulerGamma}^2\pi^2+239\,\text{eps}^4\pi^4+720\,\text{eps}\,\text{Log}[x]-792\,\text{eps}^3\pi^2\,\text{Log}[x]-288\,\text{eps}^2\,\text{Log}[x]^2+864\,\text{eps}^4\pi^2\,\text{Log}[x]^2-96\,\text{eps}^3\,\text{Log}[x]^3+192\,\text{eps}^4\,\text{Log}[x]^4+3120\,\text{eps}^3\,\text{Zeta}[3]+576\,\text{eps}^4\,\text{EulerGamma}\,\text{Zeta}[3]-4224\,\text{eps}^4\,\text{Log}[x]\,\text{Zeta}[3]\right)$$

? Series

Series[f, {x, x_0, n}] generates a power series expansion for f about the point $x = x_0$ to order $(x - x_0)^n$.
Series[f, {x, x_0, n_x}, {y, y_0, n_y}, …] successively finds series expansions with respect to x, then y, etc. ≫

Series[K[x], {eps, 0, -1}]

$$-\frac{4}{\text{eps}^4}+\frac{5\,\text{Log}[x]}{\text{eps}^3}+\frac{\frac{5\pi^2}{2}-2\,\text{Log}[x]^2}{\text{eps}^2}+\frac{1}{6\,\text{eps}}\left(-33\pi^2\,\text{Log}[x]-4\,\text{Log}[x]^3+12\pi^2\,\text{Log}[1+x]+12\,\text{Log}[x]^2\,\text{Log}[1+x]+24\,\text{Log}[x]\,\text{PolyLog}[2,-x]-24\,\text{PolyLog}[3,-x]+130\,\text{Zeta}[3]\right)+O[\text{eps}]^0$$

Series[K[x], {eps, 0, -1}] - sumresult // Simplify

$O[\text{eps}]^0$

Result by Johannes Bluemlein for 1/eps and eps^0 terms (from MOSDBO.nb)

Bluemlein =

$$\frac{1}{\text{eps}}\left(12\,z2\,\text{Log}[1+x]+2\,\text{Log}[x]^2\,\text{Log}[1+x]+4\,\text{Log}[x]\,\text{PolyLog}[2,-x]-4\,\text{PolyLog}[3,-x]\right)+$$
$$-\frac{3\,\text{EulerGamma}^2\,z2}{2}-\frac{499\,z2^2}{20}-4\,\text{EulerGamma}\,z3+4\,z3\,\text{Log}[1+x]-$$
$$20\,z2\,\text{Log}[x]\,\text{Log}[1+x]-\frac{8}{3}\text{Log}[x]^3\,\text{Log}[1+x]+6\,z2\,\text{Log}[1+x]^2+$$
$$\text{Log}[x]^2\,\text{Log}[1+x]^2+40\,z2\,\text{PolyLog}[2,-x]+2\,\text{Log}[x]^2\,\text{PolyLog}[2,-x]+$$
$$4\,\text{Log}[x]\,\text{Log}[1+x]\,\text{PolyLog}[2,-x]-24\,\text{Log}[x]\,\text{PolyLog}[3,-x]-$$
$$4\,\text{Log}[1+x]\,\text{PolyLog}[3,-x]+44\,\text{PolyLog}[4,-x]-4\,\text{PolyLog}[2,2,-x]+4\,\text{Log}[x]$$
$$\left(\frac{1}{2}\text{Log}[-x]\,\text{Log}[1+x]^2+\text{Log}[1+x]\,\text{PolyLog}[2,1+x]-\text{PolyLog}[3,1+x]+\text{Zeta}[3]\right)$$

$$-\frac{3\,\text{EulerGamma}^2\,z2}{2}-\frac{499\,z2^2}{20}-4\,\text{EulerGamma}\,z3+4\,z3\,\text{Log}[1+x]-20\,z2\,\text{Log}[x]\,\text{Log}[1+x]-$$
$$\frac{8}{3}\text{Log}[x]^3\,\text{Log}[1+x]+6\,z2\,\text{Log}[1+x]^2+\text{Log}[x]^2\,\text{Log}[1+x]^2+40\,z2\,\text{PolyLog}[2,-x]+$$
$$2\,\text{Log}[x]^2\,\text{PolyLog}[2,-x]+4\,\text{Log}[x]\,\text{Log}[1+x]\,\text{PolyLog}[2,-x]+\frac{1}{\text{eps}}$$
$$\left(12\,z2\,\text{Log}[1+x]+2\,\text{Log}[x]^2\,\text{Log}[1+x]+4\,\text{Log}[x]\,\text{PolyLog}[2,-x]-4\,\text{PolyLog}[3,-x]\right)-$$
$$24\,\text{Log}[x]\,\text{PolyLog}[3,-x]-4\,\text{Log}[1+x]\,\text{PolyLog}[3,-x]+$$
$$44\,\text{PolyLog}[4,-x]-4\,\text{PolyLog}[2,2,-x]+$$
$$4\,\text{Log}[x]\left(\frac{1}{2}\text{Log}[-x]\,\text{Log}[1+x]^2+\text{Log}[1+x]\,\text{PolyLog}[2,1+x]-\text{PolyLog}[3,1+x]+\text{Zeta}[3]\right)$$

sumresult2 = s2[[2, 1]] + Bluemlein

$$-\frac{3\,\text{EulerGamma}^2\,z2}{2}-\frac{499\,z2^2}{20}-4\,\text{EulerGamma}\,z3+4\,z3\,\text{Log}[1+x]-20\,z2\,\text{Log}[x]\,\text{Log}[1+x]-$$
$$\frac{8}{3}\text{Log}[x]^3\,\text{Log}[1+x]+6\,z2\,\text{Log}[1+x]^2+\text{Log}[x]^2\,\text{Log}[1+x]^2+40\,z2\,\text{PolyLog}[2,-x]+$$
$$2\,\text{Log}[x]^2\,\text{PolyLog}[2,-x]+4\,\text{Log}[x]\,\text{Log}[1+x]\,\text{PolyLog}[2,-x]+\frac{1}{\text{eps}}$$
$$\left(12\,z2\,\text{Log}[1+x]+2\,\text{Log}[x]^2\,\text{Log}[1+x]+4\,\text{Log}[x]\,\text{PolyLog}[2,-x]-4\,\text{PolyLog}[3,-x]\right)-$$
$$24\,\text{Log}[x]\,\text{PolyLog}[3,-x]-4\,\text{Log}[1+x]\,\text{PolyLog}[3,-x]+$$
$$44\,\text{PolyLog}[4,-x]-4\,\text{PolyLog}[2,2,-x]+4\,\text{Log}[x]$$
$$\left(\frac{1}{2}\text{Log}[-x]\,\text{Log}[1+x]^2+\text{Log}[1+x]\,\text{PolyLog}[2,1+x]-\text{PolyLog}[3,1+x]+\text{Zeta}[3]\right)+$$
$$\frac{1}{144\,\text{eps}^4}\left(-576+360\,\text{eps}^2\,\pi^2+36\,\text{eps}^4\,\text{EulerGamma}^2\,\pi^2+239\,\text{eps}^4\,\pi^4+720\,\text{eps}\,\text{Log}[x]-\right.$$
$$792\,\text{eps}^3\,\pi^2\,\text{Log}[x]-288\,\text{eps}^2\,\text{Log}[x]^2+864\,\text{eps}^4\,\pi^2\,\text{Log}[x]^2-96\,\text{eps}^3\,\text{Log}[x]^3+192\,\text{eps}^4$$
$$\left.\text{Log}[x]^4+3120\,\text{eps}^3\,\text{Zeta}[3]+576\,\text{eps}^4\,\text{EulerGamma}\,\text{Zeta}[3]-4224\,\text{eps}^4\,\text{Log}[x]\,\text{Zeta}[3]\right)$$

```
sumresult2a = sumresult2 /. z2 -> Zeta[2] /. z3 -> Zeta[3]
```

$$-\frac{1}{4}\,\text{EulerGamma}^2\,\pi^2-\frac{499\,\pi^4}{720}-\frac{10}{3}\,\pi^2\,\text{Log}[x]\,\text{Log}[1+x]-\frac{8}{3}\,\text{Log}[x]^3\,\text{Log}[1+x]+\pi^2\,\text{Log}[1+x]^2+\text{Log}[x]^2\,\text{Log}[1+x]^2+\frac{20}{3}\,\pi^2\,\text{PolyLog}[2,-x]+2\,\text{Log}[x]^2\,\text{PolyLog}[2,-x]+4\,\text{Log}[x]\,\text{Log}[1+x]\,\text{PolyLog}[2,-x]+\frac{1}{\text{eps}}\left(2\,\pi^2\,\text{Log}[1+x]+2\,\text{Log}[x]^2\,\text{Log}[1+x]+4\,\text{Log}[x]\,\text{PolyLog}[2,-x]-4\,\text{PolyLog}[3,-x]\right)-24\,\text{Log}[x]\,\text{PolyLog}[3,-x]-4\,\text{Log}[1+x]\,\text{PolyLog}[3,-x]+44\,\text{PolyLog}[4,-x]-4\,\text{PolyLog}[2,2,-x]-4\,\text{EulerGamma}\,\text{Zeta}[3]+4\,\text{Log}[1+x]\,\text{Zeta}[3]+4\,\text{Log}[x]\left(\frac{1}{2}\,\text{Log}[-x]\,\text{Log}[1+x]^2+\text{Log}[1+x]\,\text{PolyLog}[2,1+x]-\text{PolyLog}[3,1+x]+\text{Zeta}[3]\right)+\frac{1}{144\,\text{eps}^4}\left(-576+360\,\text{eps}^2\,\pi^2+36\,\text{eps}^4\,\text{EulerGamma}^2\,\pi^2+239\,\text{eps}^4\,\pi^4+720\,\text{eps}\,\text{Log}[x]-792\,\text{eps}^3\,\pi^2\,\text{Log}[x]-288\,\text{eps}^2\,\text{Log}[x]^2+864\,\text{eps}^4\,\pi^2\,\text{Log}[x]^2-96\,\text{eps}^3\,\text{Log}[x]^3+192\,\text{eps}^4\,\text{Log}[x]^4+3120\,\text{eps}^3\,\text{Zeta}[3]+576\,\text{eps}^4\,\text{EulerGamma}\,\text{Zeta}[3]-4224\,\text{eps}^4\,\text{Log}[x]\,\text{Zeta}[3]\right)$$

```
sumresult2a - K[x] // FullSimplify
```

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