Massive On-shell Planar Double Box

```
(* Time of evaluation and the version of the Mathematica kernel used *)
```

```
DateString[]
```

```
Sat 26 Sep 2015 15:59:43
```

```
$Version
```

```
9.0 for Linux x86 (64-bit) (November 20, 2012)
```

Derivation of MB representation

```
(* fin from http://prac.us.edu.pl/~gluza/ambre/examples/example7.nb *)
```

```
<< MB.m
```

```
MB 1.2

by Michal Czakon

improvements by Alexander Smirnov

more info in hep-ph/0511200

last modified 2 Jan 09
```

```
fin = -((m^2)^(z1+z6) (-s)^(z4+z7) (-t)^(-3-2 eps-z1-z4-z6-z7)
      Gamma[-z1] Gamma[-z2] Gamma[-z3] Gamma[-1 - eps - z1 - z4]
      Gamma[-1 - eps - z1 - z2 - z3 - z4] Gamma[-z4] Gamma[1 + z2 + z4] Gamma[1 + z3 + z4]
      Gamma[2 + 2 z1 + z2 + z3 + 2 z4] Gamma[-z6] Gamma[-1 - eps + z2 + z3 - z6 - z7]
      Gamma[-2 - 2 eps - z1 - z4 - z6 - z7] Gamma[-z7] Gamma[1 - z2 + z7] Gamma[1 - z3 + z7]
      Gamma[3 + 2 eps + z1 + z4 + z6 + z7] Gamma[2 - z2 - z3 + 2 z6 + 2 z7]) /
   (Gamma[-2 eps] Gamma[1 - z2] Gamma[1 - z3] Gamma[-1 - 3 eps - z1 - z4]
    Gamma[2 + z2 + z3 + 2 z4] Gamma[2 - z2 - z3 + 2 z7])
```

$$-\Big((m^2)^{z1+z6}\,(-s)^{z4+z7}\,(-t)^{-3-2\,eps-z1-z4-z6-z7}$$
$$\text{Gamma}[-z1]\,\text{Gamma}[-z2]\,\text{Gamma}[-z3]\,\text{Gamma}[-1-eps-z1-z4]$$
$$\text{Gamma}[-1-eps-z1-z2-z3-z4]\,\text{Gamma}[-z4]\,\text{Gamma}[1+z2+z4]\,\text{Gamma}[1+z3+z4]$$
$$\text{Gamma}[2+2\,z1+z2+z3+2\,z4]\,\text{Gamma}[-z6]\,\text{Gamma}[-1-eps+z2+z3-z6-z7]$$
$$\text{Gamma}[-2-2\,eps-z1-z4-z6-z7]\,\text{Gamma}[-z7]\,\text{Gamma}[1-z2+z7]\,\text{Gamma}[1-z3+z7]$$
$$\text{Gamma}[3+2\,eps+z1+z4+z6+z7]\,\text{Gamma}[2-z2-z3+2\,z6+2\,z7]\Big)\Big/$$
$$(\text{Gamma}[-2\,eps]\,\text{Gamma}[1-z2]\,\text{Gamma}[1-z3]\,\text{Gamma}[-1-3\,eps-z1-z4]$$
$$\text{Gamma}[2+z2+z3+2\,z4]\,\text{Gamma}[2-z2-z3+2\,z7])$$

```
rules = MBoptimizedRules[fin, eps → 0, {}, {eps}]
```

MBresidues::contour : contour starts and/or ends on a pole of Gamma[−eps − z1 + z2]

MBrules::norules : no rules could be found to regulate this integral

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General::stop : Further output of MBrules::norules will be suppressed during this calculation. ≫

$$\left\{\left\{eps \to -\frac{295}{384}\right\},\ \left\{z1 \to -\frac{1}{2},\ z2 \to -\frac{11}{192},\ z3 \to -\frac{3}{8},\ z4 \to -\frac{49}{384},\ z6 \to -\frac{1}{2},\ z7 \to -\frac{1}{4}\right\}\right\}$$

```
integrals = MBcontinue[fin, eps → 0, rules, Verbose → False];

ser = MBexpand[{integrals}, Exp[2 * eps EulerGamma],
   {eps, 0, -2}] // MBmerge
```

$$\Big\{\text{MBint}\Big[\Big((m^2)^{z1+z6}\,(-s)^{-2-z1-z6}\,\text{Gamma}[-z1]^3\,\text{Gamma}[1+z1]\,\text{Gamma}[-z6]^3\,\text{Gamma}[1+z6]\Big)\Big/$$
$$\Big(2\,eps^2\,t\,\text{Gamma}[-2\,z1]\,\text{Gamma}[-2\,z6]\Big),\ \Big\{\{eps \to 0\},\ \Big\{z1 \to -\frac{1}{2},\ z6 \to -\frac{1}{2}\Big\}\Big\}\Big]\Big\}$$

Derivation of sums

```
<< MBsums.v1.0.m
```

```
MBsums v1.0 by Michal Ochman

The author would like to thank Tord Riemann
for many fruitful discussions
```

```
int1 = ser[[1]] /. (m^2)^(z1+z6) (-s)^(-2-z1-z6) → (-x)^(z1+z6) (-s)^-2 (* m^2/s=x *)
```

$$\text{MBint}\left[\left((-x)^{z1+z6}\,\text{Gamma}[-z1]^3\,\text{Gamma}[1+z1]\,\text{Gamma}[-z6]^3\,\text{Gamma}[1+z6]\right)\Big/\left(2\,\text{eps}^2\,s^2\,t\,\text{Gamma}[-2\,z1]\,\text{Gamma}[-2\,z6]\right),\ \left\{\{\text{eps}\to 0\},\ \left\{z1\to -\frac{1}{2},\ z6\to -\frac{1}{2}\right\}\right\}\right]$$

```
Lk = {s → -1 / 5, x → -5, t → -1 / 10};
```

```
s1 = MBIntToSum[int1, Lk, {z1 → R, z6 → R}]
```

```
z1->L ( Re z1 < -1/2 )
z6->L ( Re z6 < -1/2 )
```

$$\left\{\text{MBsum}\left[\left((-1)^{-n1-n2}\,(-x)^{-n1-n2}\,(n1\,!)^2\,(n2\,!)^2\right)\Big/\left(2\,\text{eps}^2\,s^2\,t\,x^2\,(1+2\,n1)\,!\,(1+2\,n2)\,!\right),\ n1\geq 0\ \&\&\ n2\geq 0,\ \{n1,\ n2\}\right]\right\}$$

```
sx = Sum[((-1)^(-n1-n2) (-x)^(-n1-n2) (n1 !)^2 (n2 !)^2) / (2 eps^2 s^2 t x^2 (1 + 2 n1) ! (1 + 2 n2) !),
  {n1, 0, Infinity}, {n2, 0, Infinity}]
```

$$\frac{8\,\text{ArcSin}\left[\frac{1}{2\sqrt{x}}\right]^2}{\text{eps}^2\,s^2\,t\,(-1+4\,x)}$$

```
MBintegrate[{int1}, Lk]
```

```
Shifting contours...
Higher-dimensional integrals
Preparing MBpart1eps-2 (dim 2)
Running MBpart1eps-2
```

$$\left\{-\frac{4.6846}{\text{eps}^2},\ \left\{\frac{0.000453913}{\text{eps}^2},\ 0\right\}\right\}$$

```
Performing 0 lower-dimensional integrations with NIntegratePerforming 0 lower-dimensional :
Higher-dimensional integrals
Preparing MBpart1eps-2 (dim 2)
Running MBpart1eps-2
```

$$\left\{-\frac{4.6846}{\text{eps}^2},\ \left\{\frac{0.000453913}{\text{eps}^2},\ 0\right\}\right\}$$

```
DoAllMBSums[s1, 5, Lk] // N
```

$$-\frac{4.68459}{\text{eps}^2}$$

```
sx /. Lk // N
```

$$-\frac{4.68459}{\text{eps}^2}$$

```
(* Numerical check with V.A.Smirnov (B0, compare Eq. 5 in hep-ph/0111160v1) *)
```

```
(* x = m^2/s *)

x0 = 1 / Sqrt[1 - 4 * x]
```

$$\frac{1}{\sqrt{1-4\,x}}$$

```
mx = Log[1 - x0]
px = Log[1 + x0]
```

$$\text{Log}\left[1-\frac{1}{\sqrt{1-4\,x}}\right]$$

$$\text{Log}\left[1+\frac{1}{\sqrt{1-4\,x}}\right]$$

```
b2 = 2 * (mx - px) ^ 2
```

$$2\left(\text{Log}\left[1-\frac{1}{\sqrt{1-4\,x}}\right]-\text{Log}\left[1+\frac{1}{\sqrt{1-4\,x}}\right]\right)^2$$

```
B0 = -x0 ^ 2 / (s ^ 2 * (-t) ^ (1 + 2 * eps)) * (b2 / eps ^ 2 + rest)
```

$$-\left((-t)^{-1-2\,\text{eps}}\left(\text{rest}+1\Big/\text{eps}^2\,2\left(\text{Log}\left[1-\frac{1}{\sqrt{1-4\,x}}\right]-\text{Log}\left[1+\frac{1}{\sqrt{1-4\,x}}\right]\right)^2\right)\right)\Big/\left(s^2\,(1-4\,x)\right)$$

```
B0S = Series[B0, {eps, 0, -2}] // Normal
```

$$\left(2\left(\text{Log}\left[1-\frac{1}{\sqrt{1-4\,x}}\right]-\text{Log}\left[1+\frac{1}{\sqrt{1-4\,x}}\right]\right)^2\right)\Big/\left(\text{eps}^2\,s^2\,t\,(1-4\,x)\right)$$

```
B0S /. Lk // N
```

$$-\frac{4.68459}{\text{eps}^2}$$

```
sx /. Lk // N
```

$$-\frac{4.68459}{\text{eps}^2}$$

```
ToString[int1, InputForm, PageWidth → 60]
```

```
MBint[((-x)^(z1 + z6)*Gamma[-z1]^3*Gamma[1 + z1]*
   Gamma[-z6]^3*Gamma[1 + z6])/(2*eps^2*s^2*t*Gamma[-2*z1]*
   Gamma[-2*z6]), {{eps -> 0}, {z1 -> -1/2, z6 -> -1/2}}]
```

```
ToString[s1, InputForm, PageWidth → 60]
```

```
{MBsum[((-1)^(-n1 - n2)*(-x)^(-n1 - n2)*n1!^2*n2!^2)/
   (2*eps^2*s^2*t*x^2*(1 + 2*n1)!*(1 + 2*n2)!),
  n1 >= 0 && n2 >= 0, {n1, n2}]}
```