Present status on numerical calculation of complete 2-loop EWPOs and 3-loop prospects

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Based on collaboration with:

Ievgen Dubovyk, Ayres Freitas, Krzysztof Grzanka,
Tord Riemann, Johann Usovitsch

FCC Week 2017

Berlin, 2 June 2017

Basic references

- Ievgen Dubovyk, Ayres Freitas, JG, Tord Riemann, Johann Usovitsch
  Phys.Lett. B762 (2016) 184

  "The two-loop electroweak bosonic corrections to $\sin^2\theta^{\rm b}_{\rm eff}$"

- Tord Riemann LL16 talk, PoS LL2016 (2016) 075:
  "30 years, some 700 integrals, and 1 dessert, or:
  Electroweak two-loop corrections to the $Z\bar{b}b$ vertex", arXiv:1610.07059;
- JG LL16 talk, PoS LL2016 (2016) 034:
  "Numerical integration of massive two-loop Mellin-Barnes integrals in Minkowskian regions",
  arXiv:1607.07538

Outline

1. New results
   - 2-loop EW bosonic corrections to $\sin^2\theta_{\rm eff}^{\rm b}$
   - Basics of our numerical approach
2. Ongoing project: completion of 2-loop EWPOs
   - Calculation of bosonic 2-loop corrections for $Z \to f\bar{f}, \quad f = u, d, s, c, e, \mu, \tau$
3. Vista on the three loops ocean of EWPOs corrections
4. Summary and Outlook
   - References
5. Backup slides: details of numerical methods
   - Construction of MB integrals
   - Avoiding numerical instabilities

From Fulvio Piccinini's morning talk

Intrinsic th. uncertainties on EWPO

- from the CDR draft contribution

"Theoretical uncertainties for electroweak and Higgs-boson precision measurements at the FCC-ee"

Conveners: A. Freitas and S. Heinemeyer; Contributors: M. Beneke et al.
see talk by S. Heinemeyer

Quantity	FCC-ee	Current intrinsic error		Projected intrinsic error
M_W [MeV]	1–1.5‡	4	$(\alpha^3, \alpha^2\alpha_s)$	1
$\sin^2\theta^{\ell}_{\rm eff}$ [10^{-5}]	0.6	4.5	$(\alpha^3, \alpha^2\alpha_s)$	1.5
Γ_Z [MeV]	0.1	0.5	$(\alpha^2_{\rm bos}, \alpha^3, \alpha^2\alpha_s, \alpha\alpha_s^2)$	0.2
R_b [10^{-5}]	6	15	$(\alpha^2_{\rm bos}, \alpha^3, \alpha^2\alpha_s)$	7
R_l [10^{-3}]	1	5	$(\alpha^2_{\rm bos}, \alpha^3, \alpha^2\alpha_s)$	1.5

‡The pure experimental precision on M_W is ~ 0.5 MeV [3].

- with present and conceivable loop technology, the intrinsic th. uncertainties will be at the same level of the experimental errors
- new calculation methods should be introduced

see talk by J. Gluza

see e.g. the recent review on multi-loop integrals, A. Freitas, Prog. Part. Nucl. Phys. 90 (2016) 201

Pseudo-observables, an example: $d\sigma/d\cos\theta\ (e^+e^- \to \bar{b}b)$

Close to the Z-boson peak and assuming Born-like v, a couplings:

$$\frac{d\sigma}{d\cos\theta} \sim G_F^2 \left| \frac{s}{s - M_Z^2 + iM_Z\Gamma_Z} \right|^2 \times \left[(a_e^2+v_e^2)(a_b^2+v_b^2)(1+\cos^2\theta)\ \ (2a_e v_e)(2a_b v_b)(2\cos\theta) \right]$$

Factorization (of weak couplings):

Symmetric integration over $\cos\theta$

$$\sigma_T = \int_{-1}^{1} d\cos\theta \frac{d\sigma}{d\cos\theta} \sim \left| \frac{s}{s - M_Z^2 + iM_Z\Gamma_Z} \right|^2 G_F(a_e^2+v_e^2)\ \ \mathbf{G_F(a_b^2+v_b^2)}$$

Anti-symmetric integration over $\cos\theta$

$$A_{F-B} = \frac{\left[\int_0^1 d\cos\theta - \int_{-1}^0 d\cos\theta\right] \frac{d\sigma}{d\cos\theta}}{\sigma_T} \sim \overbrace{\frac{2a_e v_e}{a_e^2+v_b^2}}^{A_e}\ \ \overbrace{\mathbf{\frac{2a_b v_b}{a_b^2+v_b^2}}}^{A_b}$$

Pseudo-observable A_b

$$A_{\rm b} = \frac{2\Re e \frac{v_b}{a_b}}{1+\left(\Re e\frac{v_b}{a_b}\right)^2} = \frac{1-4|Q_b|\sin^2\theta^{\rm b}_{\rm eff}}{1-4|Q_b|\sin^2\theta^{\rm b}_{\rm eff}+8Q_b^2(\sin^2\theta^{\rm b}_{\rm eff})^2}$$

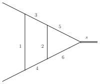
Definition of the effective weak mixing angle

$$\sin^2\theta^{\rm b}_{\rm eff} = \frac{1}{4|Q_b|}\left(1-\Re e\frac{v_b}{a_b}\right)$$

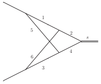
- Vertex form factor

$$V_\mu^{Zb\bar{b}} = \gamma_\mu[v_b(s) - a_b(s)\gamma_5] = \cdots + \overbrace{\underbrace{D_{\rm planar} + D_{\rm non-planar}}_{\rm planar,non-planar}}^{\rm fermionic,bosonic} + \cdots$$

fermionic,bosonic



+



planar,non−planar

Our results: Effective weak mixing angle $\sin^2\theta^{\rm b}_{\rm eff}$

- The standard model prediction for the effective weak mixing angle can be written as

$$\sin^2\theta^{\rm b}_{\rm eff} = \left(1 - \frac{M_W^2}{M_Z^2}\right)(1 + \Delta\kappa_{\rm b})$$

- The bosonic electroweak two-loop corrections amount to

$$\boxed{\Delta\kappa_{\rm b}^{(\alpha^2,{\rm b}os)} = -0.9855 \times 10^{-4}}$$

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Collection of radiative corrections: full stabilization at 10^{-4}!

Order	Value $[10^{-4}]$	Order	Value $[10^{-4}]$
α	468.945	$\alpha_t^2\alpha_s$	1.362
$\alpha\alpha_s$	-42.655	α_t^3	0.123
$\alpha_t\alpha_s^2$	-7.074	$\alpha^2_{\rm ferm}$	3.866
$\alpha_t\alpha_s^3$	-1.196	$\alpha^2_{\rm bos}$	**-0.986**

± 0.001

Table: *Comparison of different orders of radiative corrections to* $\Delta\kappa_{\rm b}$.

Input Parameters: M_Z, Γ_Z, M_W, Γ_W, M_H, m_t, α_s *and* $\Delta\alpha$

- one-loop contributions [Akhundov, Bardin, Riemann, 1986] [Beenakker, Hollik, 1988]
- two-loop fermionic contributions [Awramik, Czakon, Freitas, Kniehl, 2009]
- two-loop bosonic contributions [Dubovyk, Freitas, JG, Riemann, Usovitsch, 2016]

Partial higher-order corrections

$\mathcal{O}(\alpha_{\rm t}\alpha_{\rm s}^2)$	Avdeev: 1994,Chetyrkin: 1995
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$\mathcal{O}(\alpha^2\alpha_{\rm t})$ and $\mathcal{O}(\alpha_{\rm t}^3)$	vanderBij: 2000,Faisst: 2003

Currently most precise prediction for $\sin^2\theta^{\rm b}_{\rm eff}$

$$\sin^2\theta^{\rm b}_{\rm eff} = s_0 + d_1\,L_H + d_2\,L_H^2 + d_3\,\Delta_\alpha + d_4\,\Delta_t + d_5\,\Delta_t^2 + d_6\,\Delta_t\,L_H + d_7\,\Delta_{\alpha_{\rm s}} + d_8\,\Delta_t\,\Delta_{\alpha_{\rm s}} + d_9\,\Delta_Z \qquad (1)$$

$$L_H = \log\left(\frac{M_{\rm H}}{125.7{\rm Ge}V}\right), \quad \Delta_t = \left(\frac{m_{\rm t}}{173.2{\rm Ge}V}\right)^2 - 1, \quad \Delta_Z = \frac{M_{\rm Z}}{91.1876{\rm Ge}V} - 1,$$
$$\Delta_\alpha = \frac{\Delta\alpha}{0.0059} - 1, \quad \Delta_{\alpha_{\rm s}} = \frac{\alpha_{\rm s}}{0.1184} - 1. \qquad (2)$$

$$s_0 = 0.232704, \quad d_1 = 4.723\times10^{-4}, \quad d_2 = 1.97\times10^{-4}, \quad d_3 = 2.07\times10^{-2},$$
$$d_4 = -9.733\times10^{-4}, \quad d_5 = 3.93\times10^{-4}, \quad d_6 = -1.38\times10^{-4},$$
$$d_7 = 2.42\times10^{-4}, \quad d_8 = -8.10\times10^{-4}, \quad d_9 = -0.664. \qquad (3)$$

- $M_{\rm W}$ is calculated from the Fermi constant G_μ [Awramik, et al., 2004]
- The deviations to the full calculation amount to average (maximal) 2×10^{-7} (1.3×10^{-6}), in the input parameter ranges.

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Side but important remarks

$\mathcal{O}(\alpha^2_{\rm bos})$ correction expected to be not the dominant source of intrinsic uncertainty (up to two-loops),

BUT:

(a) We know it just now, for $Z \to b\bar{b}$ case;
(a') Other known corrections have the same (small) size;

(b) If the dominant 3-loop corrections become available, $\mathcal{O}(\alpha^2_{\rm bos})$ correction will become very important.

(c) We are establishing a new technique $\longrightarrow$ further applications possible.

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Direct numerical integrations in Minkowskian regions. Two general methods beyond NLO.

Sector decomposition (SD)

FIESTA 3 [A.V.Smirnov, 2014]
SecDec 3 [2012], pySecDec [2017] [S. Borowka, G. Heinrich, S. Jahn, P. Jones, M. Kerner, J. Schlenk, T. Zirke]

- NICODEMOS, ver 2.0 [A. Freitas]

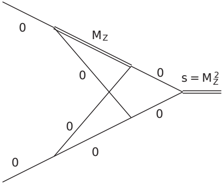
Now: The Mellin-Barnes (MB) method.

Toolbox: AMBRE/PlanarityTest/MBtools/MBnumerics/CUBA
- see References list on slide [24]

Two steps (automatic construction and numerical evaluation):

1. AMBRE [JG, K.Kajda, T.Riemann (planar, ver.2.2), JG, E.Dubovyk, T.Riemann (non-planar,ver3.0)], PlanarityTest [K.Bielas, E.Dubovyk, JG, T. Riemann]
2. MBnumerics [E. Dubovyk, J. Usovitsch, T. Riemann] - a completely new software !

First example: MBnumerics - fine, SD - more difficult



Euclidean results (constant part):

Analytical :	$-\mathbf{0.49661983}06057021$
MB(Vegas) :	-0.4969417442183914
MB((**Cuhre**) :	$-\mathbf{0.49661983}13219404$ ←
FIESTA :	-0.4966184488196595
SecDec :	-0.4966192150541896

Minkowskian results (constant part):

2016 →	Analytical :	$\mathbf{0.77859960}8979684 + \mathbf{4.12351}2593396311 \cdot i$
	MBnumerics :	$\mathbf{0.77859960}8324769 + \mathbf{4.12351}2600516016 \cdot i$
	MB(Vegas) :	big error
	MB(Cuhre) :	NaN
	FIESTA :	big error
2017 →	SecDec :	$\mathbf{0.78} + \mathbf{4.13} \cdot i$ (pySecDec)

MB and SD methods are very much complementary!

- MB works well for hard threshold, on-shell cases, not many internal masses (more IR), SD is useful for integrals with internal masses.
- Discussed already at: J. Gluza, K. Kajda, T. Riemann and V. Yundin *"Numerical Evaluation of Tensor Feynman Integrals in Euclidean Kinematics"*, EPJC **71** (2011) 1516; [arXiv:1010.1667 [hep-ph]]

Now realized in real application for EWPOs:

10^{-8} accuracy achieved for **any** self-energy and vertex Feynman integral with one of the methods.

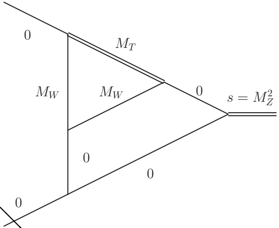
Cuhre [Thomas Hahn] - not a Monte Carlo, crucial for MB with dim$\leq$5.

Must be stressed:

We had to solve several problems. More importantly:

- **Minimal** MB-representations (new automatic approach to non-planar diagrams);
- Numerical **stabilities, accuracy and speed** of calculation for **many scales**;
- Numerical stabilities, accuracy and speed of calculation for **Minkowskian region**
- For details see backup slides

Second example. One of the most difficult IR-divergent integrals with 2 scales for SD



1st problem:
Integrating tails $z_i \to \infty$, $s = M_Z^2$!

2nd problem: Γ 's.

$$\int dz_1 \int dz_2 \int dz_3 \; (-s)^{-2-2\epsilon} \left(-\frac{\mathbf{s}}{\mathbf{MT^2}}\right)^{-z_2} \left(-\frac{\mathbf{s}}{\mathbf{MW^2}}\right)^{-z_1} \Gamma[-z_1]\Gamma[-z_2]$$

$$\times \quad \frac{\Gamma[-z_3]\Gamma[-1-2\epsilon-z_1-z_2]\Gamma[-\epsilon-z_1-z_2]\Gamma[2+2\epsilon+z_1+z_2]}{\Gamma[-3\epsilon-z_1-z_2]\Gamma[1-2\epsilon-z_1-z_2]\Gamma[1-z_3]\Gamma[-2\epsilon-2z_1-2z_2-z_3]}$$

$$\times \quad \Gamma[-2\epsilon-z_1-2z_2-z_3]\Gamma[-1-2\epsilon-z_1-z_2-z_3]$$

$$\times \quad \Gamma[-\epsilon-z_1-z_2-z_3]\Gamma[1+z_2+z_3]\Gamma[1+\epsilon+z_1+z_2+z_3]$$

Solutions: see Backup Slides and MBnumerics.m

One of the most difficult IR-divergent integrals with 2 scales, cont'd.

For MB it is much easier and better.

MBnumerics.m 2016-04-21 Johann Usovitsch

```
=
    1.541402128186602 + 0.248804198197504*I
+
   0.1236145994284665 9 - 1.0610332704387688 *I * eps^-1
+
  -0.3377373795505797 0 + 3.6*10^-17*I *eps^-2
```

Time needed **43 min.**

SecDec = 1.541 + 0.2487*I

```
+
     0.123615 - 1.06103*I *eps^-1
+
    -0.3377373796 - 5*10^-10*I*eps^-2
```

Time needed **24 hours**

Types of numerical improvements in MBnumerics.m

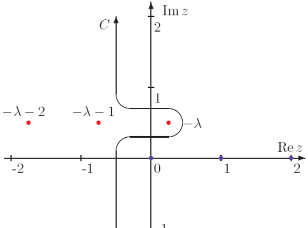
THE BASIC METHOD:

Contour shifts - new, effective for n-dim MB integrations

Step by step with more techniques (needed for speed and precision):

(recent summary: http://arxiv.org/abs/arXiv:1704.02288)

(a) Contour rotations (deformations);
(b) Mappings;
(c) Variables transformations, e.g. $\{z_1, z_2\} \to \{z_1 - z_2, z_2\}$;



Contours: shifts, deformations

Ongoing project: $Z \to f\bar{f}, f = u, d, s, c, e, \mu, \tau$

- Comparing to $Z \to b\bar{b}$, there are no top quarks inside "bosonic" diagrams - one scale less. **but:**
- More massless particles involved - some diagrams with more complicated structure of singularities **(spurious divergencies)**.
- The goal is to complete EWPO at 2 loops:

$$
\begin{array}{c}
\{\Gamma_{Zff}, \Gamma_{Zbb}\} \longrightarrow \Gamma_{Z_{\text{tot}}} \\
R_\ell \\
R_{c,b} \\
A^f_{\text{FB}} \\
A^f_{\text{LR}}
\end{array}
$$

Even then:

FCCee demands more ... (see F. Piccinini talk)

Look into the future. Bookkeeping with three loops

$Z \to b\bar{b}$			
Number of topologies	1 loop	2 loops	3 loops
	1	$14 \to^{\mathbf{A}} 7 \to^{\mathbf{B}}$ **5**	$211 \to^{\mathbf{A}} 84 \to^{\mathbf{B}}$ **50**
Number of diagrams	15	$2383 \to^{\mathbf{A,B}}$ **1114**	$490387 \to^{\mathbf{A,B}}$ **120187**
Fermionic loops	0	371	116091
Bosonic loops	15	2012	374296
Planar	1T/15D	13T/2250D	186T/426753D
Non-planar	0	1T/133D	25T/63634D

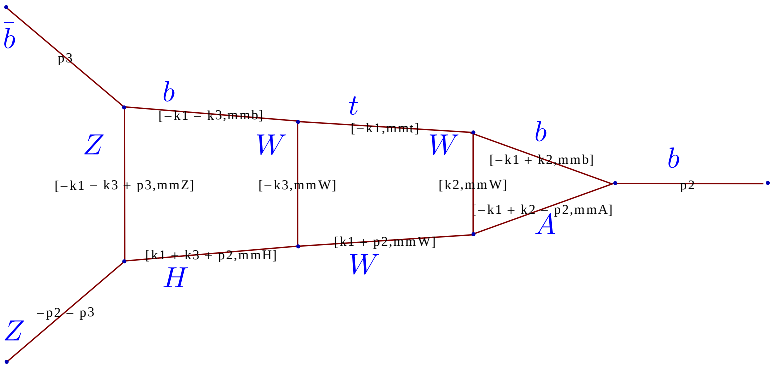
$Z \to e^+e^-, \ldots$			
Number of topologies	1 loop	2 loops	3 loops
	1	$14 \to^{\mathbf{A}} 7 \to^{\mathbf{B}}$ **5**	$211 \to^{\mathbf{A}} 84 \to^{\mathbf{B}}$ **50**
Number of diagrams	14	$2012 \to^{\mathbf{A,B}}$ **880**	$397690 \to^{\mathbf{A,B}}$ **91271**
Fermionic loops	0	301	92397
Bosonic loops	14	1711	305293
Planar	1	13	186
Non-planar	0	1	25

A - no tadpoles, no product of lower loops,
(B) - symmetry included

Genuine virtual loops (aITALC, qgraf, FeynArts)

A complete zoo of heavy particles m_t, m_W, m_Z, m_H

MB: ϵ^0**[8-dim]**, $1/\epsilon$**[7-dim]**; SD: ϵ^0**[8-dim]** , $1/\epsilon$**[7-dim]**;

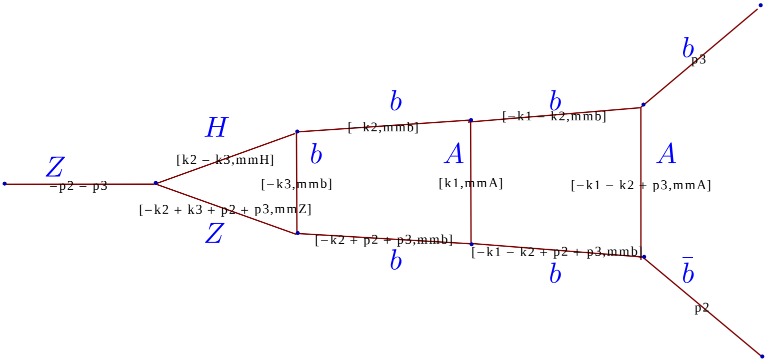


At 2-loops up to three dimensionless parameters (now 4):

$$\left\{ \frac{M_H^2}{M_Z^2}, \frac{M_W^2}{M_Z^2}, \frac{m_t^2}{M_Z^2}, \frac{(M_Z + i\varepsilon)^2}{M_Z^2} \right\}$$

Less masses

MB: ϵ^0**[6-dim]**, $1/\epsilon$**[5-dim]**,...,$1/\epsilon^4$**[2-dim]**; SD: ϵ^0**[8-dim]**,...,$1/\epsilon^4$**[4-dim]**



New improved AMBREv1.3, AMBREv2.1

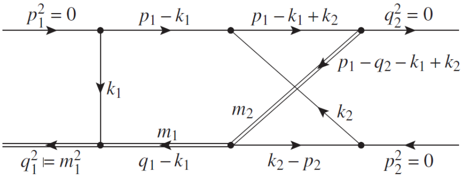
Non-planar AMBREv3.0 will be tuned, if possible

Next years?

Permanent and substantial progress in loop calculations:

- Mellin-Barnes (2016): AMBRE3, MBnumerics.m, and work in progress
  Sector Decomposition (2017): pySecDec
- Differential equations 2013 - Henn - proposal for uniform system of equations
  Packages (towards solutions of master integrals with many scales):
  Prausa (2017): "epsilon: A tool to find a canonical basis of master integrals"
  Meyer (2017): "Algorithmic transformation of multi-loop master integrals to a canonical basis with CANONICA"
  Gituliar, Magerya (2017): "Fuchsia: a tool for reducing differential equations for Feynman master integrals to epsilon form"
- Maierhoefer, Usovitsch, Uwer (2017): "Kira - A Feynman Integral Reduction Program"

Example: IBP reductions, Maierhoefer, Usovitsch, Uwer, 1705.05610



$s_{\max}$	T_{pyRed}	T_{Kira}	T_{Reduze}	$\frac{T_{\text{pyRed}}}{T_{\text{Kira}}}$	$\frac{T_{\text{Reduze}}}{T_{\text{Kira}}}$
1	2.8 s	90 sec	4 h	0.03	160
2	9.8 s	6.6 min	13.3 h	0.02	121
3	28 s	43 min	2 d	0.01	67
4	67 s	2.4 h	7 d	0.007	70

⋆ **So far (2-loops) we calculated tensor integrals directly** (no IBP reductions)

However...

Although analytical approaches are nice and make important progress, development of numerical methods for completion realistic electroweak problems with several or many scales seems to be a right way of acting.

Conclusions

- MB is a mighty method. With MB-suite at hand, we are ready to complete EWPOs at 2 loops
- A universal numerical package for arbitrary $\geq$ 2-loop massive IR-divergent Feynman integrals in the Minkowskian kinematics has to integrate both methods, SD and MB.
- On larger time scale - 3 loops prospect is a beautiful place for theoretical work
- and still much must be done on tools and methods to get it beyond the present status

Partial higher-order corrections

$\mathcal{O}(\alpha_t \alpha_s^2)$	Avdeev: 1994,Chetyrkin: 1995
$\mathcal{O}(\alpha_t \alpha_s^3)$	Schroder: 2005,Chetyrkin: 2006,Boughezal: 2006
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Detail references below

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Mellin-Barnes with related packages and papers I

[1] M. Czakon (MB, MBasymptotics),
D. Kosower (barnesroutines),
A. Smirnov, V. Smirnov (MBresolve),
K. Bielas, I. Dubovyk, J. Gluza, K. Kajda, T. Riemann (AMBRE),

MBtools webpage, https://mbtools.hepforge.org/.

[2] AMBRE webpage with additional software: PlanarityTest, MBSums, KinematicsGen

http://prac.us.edu.pl/~gluza/ambre.

[3] J. Gluza, K. Kajda, T. Riemann, AMBRE - a Mathematica package for the construction of Mellin-Barnes representations for Feynman integrals, Comput. Phys. Commun. 177 (2007) 879–893. arXiv:0704.2423, doi:10.1016/j.cpc.2007.07.001.

[4] J. Gluza, K. Kajda, T. Riemann, V. Yundin, Numerical Evaluation of Tensor Feynman Integrals in Euclidean Kinematics, Eur. Phys. J. C71 (2011) 1516. arXiv:1010.1667, doi:10.1140/epjc/s10052-010-1516-y.

[5] K. Bielas, I. Dubovyk, J. Gluza, T. Riemann, Some Remarks on Non-planar Feynman Diagrams, Acta Phys. Polon. B44 (2013) 2249–2255. arXiv:1312.5603, doi:10.5506/APhysPolB.44.2249.

[6] I. Dubovyk, J. Gluza, T. Jelinski, T. Riemann and J. Usovitsch, New prospects for the numerical calculation of Mellin-Barnes integrals in Minkowskian kinematics, arXiv:1704.02288.

[7] M. Czakon, Automatized analytic continuation of Mellin-Barnes integrals, Comput. Phys. Commun. 175 (2006) 559–571. arXiv:hep-ph/0511200, doi:10.1016/j.cpc.2006.07.002.

Mellin-Barnes with related packages and papers II

[8] A. Smirnov, V. Smirnov, On the Resolution of Singularities of Multiple Mellin-Barnes Integrals, Eur. Phys. J. C62 (2009) 445–449. arXiv:0901.0386, doi:10.1140/epjc/s10052-009-1039-6.

[9] T. Hahn, CUBA: A library for multidimensional numerical integration, Comput. Phys. Commun. 168 (2005) 78–95. arXiv:hep-ph/0404043, doi:10.1016/j.cpc.2005.01.010.

[10] I. Dubovyk, A. Freitas, J. Gluza, T. Riemann and J. Usovitsch, "The two-loop electroweak bosonic corrections to $\sin^2\theta^{\rm b}_{\rm eff}$", Phys. Lett. B **762** (2016) 184. doi:10.1016/j.physletb.2016.09.012

[11] I. Dubovyk, J. Gluza, T. Riemann, J. Usovitsch, Numerical integration of massive two-loop Mellin-Barnes integrals in Minkowskian regions, PoS LL2016 (2016) 034. arXiv:1607.07538.

[12] I. Dubovyk, A. Freitas, J. Gluza, T. Riemann, J. Usovitsch, 30 years, some 700 integrals, and 1 dessert, or: Electroweak two-loop corrections to the $Z\bar{b}b$ vertex, PoS LL2016 (2016) 075. arXiv:1610.07059.

[13] A. Freitas, Y.-C. Huang, On the Numerical Evaluation of Loop Integrals With Mellin-Barnes Representations, JHEP 04 (2010) 074. arXiv:1001.3243, doi:10.1007/JHEP04(2010)074.

[14] A. Freitas, Y.-C. Huang, Electroweak two-loop corrections to $sin^2\theta^{b\bar{b}}_{\rm eff}$ and R_b using numerical Mellin-Barnes integrals, JHEP 1208 (2012) 050. arXiv:1205.0299, doi:10.1007/JHEP08(2012)050.

[15] J. Gluza, T. Jelinski and D. A. Kosower, Phys. Rev. D **95** (2017) no.7, 076016 doi:10.1103/PhysRevD.95.076016 [arXiv:1609.09111 [hep-ph]].

[16] M. Ochman, T. Riemann, MBsums - a Mathematica package for the representation of Mellin-Barnes integrals by multiple sums, Acta Phys. Polon. B46 (2015) 2117. arXiv:1511.01323, doi:10.5506/APhysPolB.46.2117.

References for EWPOs corrections I

$\mathcal{O}(\alpha\alpha_s)$ QCD corrections:

[1] A. Djouadi, C. Verzegnassi, Virtual very heavy top effects in LEP/SLC precision measurements, Phys. Lett. B195 (1987) 265–271. doi:10.1016/0370-2693(87)91206-8.

[2] A. Djouadi, $O(\alpha\alpha_s)$ vacuum polarization functions of the standard model gauge bosons, Nuovo Cim. A100 (1988) 357. doi:10.1007/BF02812964.

[3] B. A. Kniehl, Two loop corrections to the vacuum polarizations in perturbative QCD, Nucl. Phys. B347 (1990) 86–104. doi:10.1016/0550-3213(90)90552-O.

[4] B. A. Kniehl, A. Sirlin, Dispersion relations for vacuum polarization functions in electroweak physics, Nucl. Phys. B371 (1992) 141–148. doi:10.1016/0550-3213(92)90232-Z.

[5] A. Djouadi, P. Gambino, Electroweak gauge bosons selfenergies: Complete QCD corrections, Phys. Rev. D49 (1994) 3499–3511, Erratum: Phys. Rev. D53 (1996) 4111. arXiv:hep-ph/9309298, doi:10.1103/PhysRevD.49.3499,10.1103/PhysRevD.53.4111.

[6] J. Fleischer, O. Tarasov, F. Jegerlehner, P. Raczka, Two loop $\mathcal{O}(\alpha_s G_\mu m_t^2)$ corrections to the partial decay width of the Z^0 into $b\bar{b}$ final states in the large top mass limit, Phys. Lett. B293 (1992) 437–444. doi:10.1016/0370-2693(92)90909-N.

[7] G. Buchalla, A. Buras, QCD corrections to the $\bar{s}dZ$ vertex for arbitrary top quark mass, Nucl. Phys. B398 (1993) 285–300. doi:10.1016/0550-3213(93)90110-B.

[8] G. Degrassi, Current algebra approach to heavy top effects in $Z \to b + \bar{b}$, Nucl. Phys. B407 (1993) 271–289. arXiv:hep-ph/9302288, doi:10.1016/0550-3213(93)90058-W.

[9] K. Chetyrkin, A. Kwiatkowski, M. Steinhauser, Leading top mass corrections of order $O(\alpha\alpha_s m_t^2/M_W^2)$ to partial decay rate $\Gamma(Z \to b\bar{b})$, Mod. Phys. Lett. A8 (1993) 2785–2792. doi:10.1142/S0217732393003172.

References for EWPOs corrections II

[10] A. Czarnecki, J. H. Kühn, Nonfactorizable QCD and electroweak corrections to the hadronic Z boson decay rate, Phys. Rev. Lett. 77 (1996) 3955–3958. arXiv:hep-ph/9608366, doi:10.1103/PhysRevLett.77.3955.

[11] R. Harlander, T. Seidensticker, M. Steinhauser, Complete corrections of order $O(\alpha\alpha_s)$ to the decay of the Z boson into bottom quarks, Phys. Lett. B426 (1998) 125–132. arXiv:hep-ph/9712228, doi:10.1016/S0370-2693(98)00220-2.

Partial higher order corrections of order $\mathcal{O}(\alpha_t\alpha_s^2)$:

[1] L. Avdeev, J. Fleischer, S. Mikhailov, O. Tarasov, $O(\alpha\alpha_s^2)$ correction to the electroweak ρ parameter, Phys. Lett. B336 (1994) 560–566, Erratum: Phys. Lett. B349 (1995) 597. arXiv:hep-ph/9406363, doi:10.1016/0370-2693(94)90573-8.

[2] K. Chetyrkin, J. H. Kühn, M. Steinhauser, Corrections of order $O(G_F M_t^2 \alpha_s^2)$ to the ρ parameter, Phys. Lett. B351 (1995) 331–338. arXiv:hep-ph/9502291, doi:10.1016/0370-2693(95)00380-4.

Partial higher order corrections of order $\mathcal{O}(\alpha_t\alpha_s^3)$:

[1] Y. Schröder, M. Steinhauser, Four-loop singlet contribution to the ρ parameter, Phys. Lett. B622 (2005) 124–130. arXiv:hep-ph/0504055, doi:10.1016/j.physletb.2005.06.085.

[2] K. G. Chetyrkin, M. Faisst, J. H. Kühn, P. Maierhofer, C. Sturm, Four-loop QCD corrections to the ρ parameter, Phys. Rev. Lett. 97 (2006) 102003. arXiv:hep-ph/0605201, doi:10.1103/PhysRevLett.97.102003.

[3] R. Boughezal, M. Czakon, Single scale tadpoles and $O(G_F m_t^2 \alpha_s^3)$ corrections to the ρ parameter, Nucl. Phys. B755 (2006) 221–238. arXiv:hep-ph/0606232.

Partial higher order corrections of orders $\mathcal{O}(\alpha_t^2\alpha_s)$ and $\mathcal{O}(\alpha_t^3)$:

References for EWPOs corrections III

[1] J. J. van der Bij, K. G. Chetyrkin, M. Faisst, G. Jikia, T. Seidensticker, Three loop leading top mass contributions to the ρ parameter, Phys. Lett. B498 (2001) 156–162. arXiv:hep-ph/0011373, doi:10.1016/S0370-2693(01)00002-8.

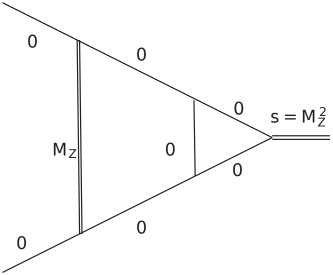
[2] M. Faisst, J. H. Kühn, T. Seidensticker, O. Veretin, Three loop top quark contributions to the ρ parameter, Nucl. Phys. B665 (2003) 649–662. arXiv:hep-ph/0302275, doi:10.1016/S0550-3213(03)00450-4.

Backup slides

Basic problem: Steping up from Euclidean to direct calculation in Minkowskian kinematics

$$\frac{1}{(-)p^2 - m^2} \longrightarrow singularities \longrightarrow \frac{1}{(-)p^2 - m^2 + i\delta}$$

Resonance: $s = M_Z^2, \quad s = -M_Z^2$



Step 1

Construction of MB integrals

http://us.edu.pl/∼gluza/ambre/

Mellin-Barnes representations in HEP - method

- "Om definita integraler", R. H. Mellin, Acta Soc. Sci. Fenn. 20(7), 1 (1895), "The theory of the gamma function", E. W. Barnes Messenger Math. 29(2), 64 (1900).

$$mathematics \longrightarrow \frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \Gamma(\lambda+z)\Gamma(-z)\frac{B^z}{A^{\lambda+z}}$$

$$physics \longrightarrow \frac{1}{(p^2-m^2)^a} = \frac{1}{\Gamma(a)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \Gamma(a+z)\Gamma(-z)\frac{(m^2)^z}{(p^2)^{a+z}}$$

It is recursive $\Longrightarrow$ multidimensional complex integrals.

$$\int\limits_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty} dz_1 \int\limits_{-\frac{2}{3}-i\infty}^{-\frac{2}{3}+i\infty} dz_2 \left(\frac{-s}{M_Z^2}\right)^{-z_1} \frac{\Gamma[-z_1]^3\Gamma[1+z_1]\Gamma[z_1-z_2]\Gamma[-z_2]^3\Gamma[1+z_2]\Gamma[1-z_1+z_2]}{s \quad \Gamma[1-z_1]^2\Gamma[-z_1-z_2]\Gamma[1+z_1-z_2]}$$

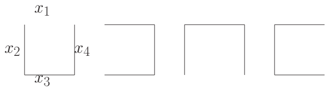
Overlaped integrals

Multiloop Feynman diagrams, general MB integrals

$$\frac{1}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} \to \int \prod_{j=1}^{N} dx_j \; x_j^{n_j - 1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

$$N_\nu = n_1 + \ldots + n_N$$

The functions U and F are called graph or Symanzik polynomials.

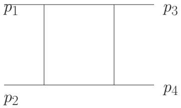


Trees contributing to the polynomial U for the square diagram

$$\mathbf{U} \;=\; \mathbf{x_1 + x_2 + x_3 + x_4} \quad ! \; \textit{1-loop} \longrightarrow 1$$

$$\mathbf{F} \;=\; \mathbf{t \cdot x_1 x_3 + s \cdot x_2 x_4}$$

2 – trees contributing to the polynomial F for the square diagram



Cuts of internal lines such that:

- U: (i) every vertex is still connected to every other vertex by a sequence of uncut lines; (ii) no further cuts without violating (i)
- F: (iii) divide the graph into two disjoint parts such that within each part (i) and (ii) are obeyed and such that at least one external

Dimension of MB integrals depends on

Step 2

Numerics of MB integrals

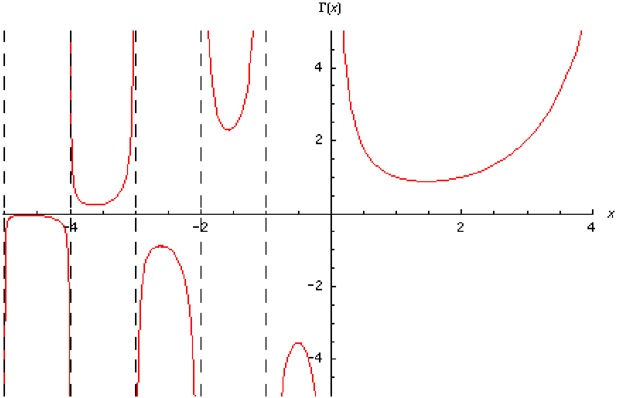
http://mbtools.hepforge.org/

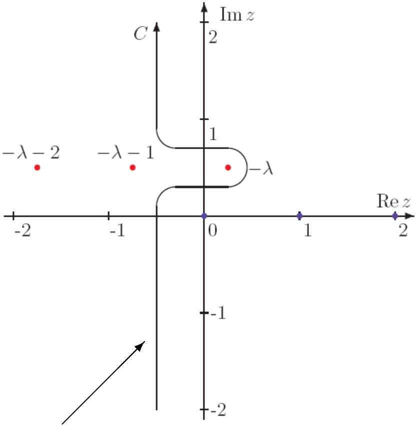
Gamma function: Singularities in the complex plane

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

$$\int dz \Gamma[z+\lambda]$$

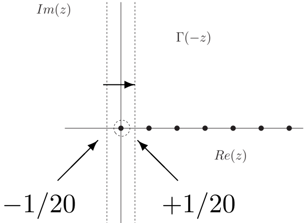
SINGULARITIES REGULAR





Contours: shifts, deformations

```
(* shifting contours *)

In[203]:=
sim = Gamma[-z]
Out[203]=
Gamma[-z]

In[227]:=
Sum[-Residue[Gamma[-z], {z, n}], {n, 0, 100}] // N
Out[227]=
0.367879

In[226]:=
n1 = NIntegrate[
  1 / (2 Pi) sim /. z → -1 / 20 + I y, {y, -10, 10}]
Out[226]=
0.367879 + 0. ⅈ

In[230]:=
n2 = NIntegrate[
  1 / (2 Pi) sim /. z → 1 / 20 + I y, {y, -10, 10}]
Out[230]=
-0.632121 + 0. ⅈ

In[231]:=
n2 - n1
Out[231]=
-1. + 0. ⅈ

In[232]:=
Residue[sim, {z, 0}]
Out[232]=
-1
```



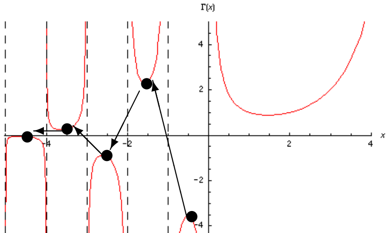
Two basic observations for shifting z follows

$$\int dz_1 \ldots dz_k \ldots I(\ldots, Re[z_k] + n + Im[z_k], \ldots) \qquad I_{orig}$$

$$= \quad Residue[\int dz_1 \ldots \not{dz_k} \ldots I]_{Re[z_k]+n} \qquad I_{Res}$$

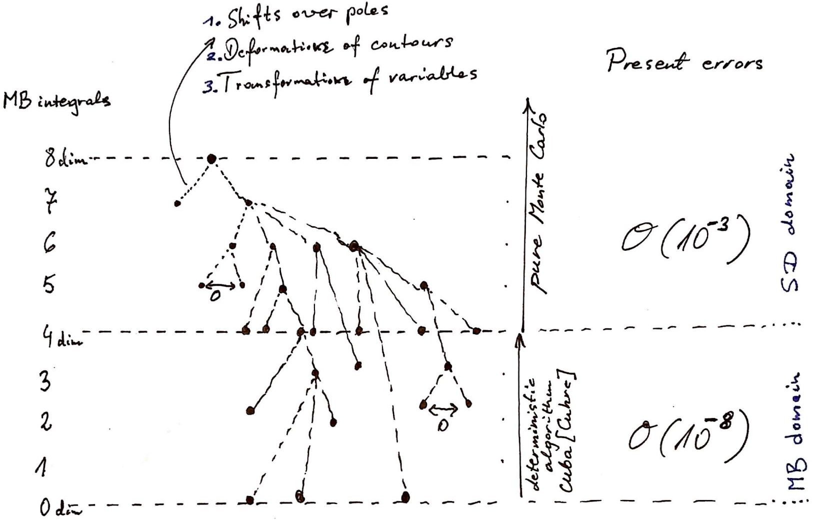
$$+ \quad \int dz_1 \ldots dz_k \ldots I(\ldots, Re[z_k] + (n+1) + Im[z_k], \ldots) \qquad I_{new}$$

1. Residues **lower** dimensionality of original MB integrals.
2. Integral after passing a pole (proper shifts) **can be made smaller**.



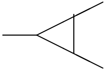
Top-bottom approach to evaluation of multidimensional MB integrals

MBnumerics.m - I. Dubovyk, J. Usovitsch, T. Riemann



BASIC PROBLEMS in Minkowski kinematics

I. Bad oscillatory behavior of integrands;
II. Fragile stability for integrations over products and ratios of Γ functions.



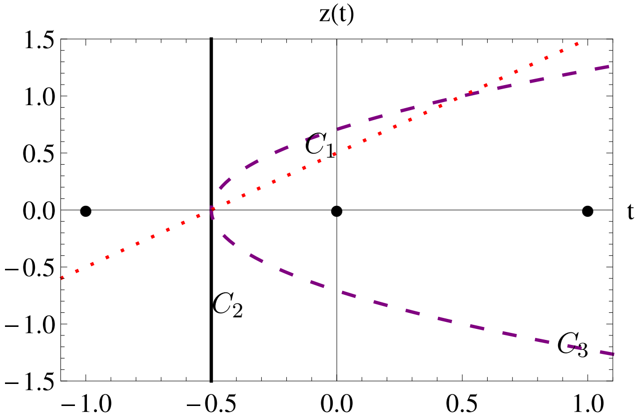
$$V(s) = \frac{e^{\epsilon\gamma_E}}{i\pi^{(4-2\epsilon)/2}} \int \frac{d^{(4-2\epsilon)}k}{[(k+p_1)^2-m^2][k^2][(k-p_2)^2-m^2]}$$

$$= \frac{V_{-1}(s)}{\epsilon} + V_0(s) + \cdots ,$$

$$V_{-1}(s)|_{m=1} = -\frac{1}{2s} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{dz}{2\pi i} \underbrace{(-s)^{-z}}_{\textbf{Problem I}} \overbrace{\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma(-2z)}}^{\textbf{Problem II}}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} = \frac{2\arcsin(\sqrt{s}/2)}{\sqrt{4-s}\sqrt{s}},$$

Contour deformations



$$z(t) = x_0 + it \quad : \qquad V_{-1}^{C_1}(s) \;=\; \int_{-\infty}^{+\infty} (i)\; dt\; J[z(t)];$$

$s = 2,\ z(t) = \Re[-1/2] + i\,y, \quad y \in (-a, +a)$

$$V_{-1}(2)|_{\text{analyt.}} = \mathbf{0.785398163397448309} = \frac{\pi}{\mathbf{4}}$$

$$V_{-1}(2)|_{\text{Pantis}}^{MB.m} = 0.7925 - \cancel{0.0225\, i}$$

$$V_{-1}(2)|_{C_1,\, a=15} = 0.7548660085063523 - \cancel{0.2299852588200015\, i}$$

$$V_{-1}(2)|_{C_1,\, a=10^2} = 0.73479313088852537844 + \cancel{0.074901423602937676597\, i}$$

$$V_{-1}(2)|_{C_1,\, a=10^3} = 0.84718185073531076915 - \cancel{0.094865760649354977853\, i}$$

$$V_{-1}(2)|_{C_1,\, a=10^4} = 4.4574554985139977188 + \cancel{4.5139812364645122275\, i}$$

✓ $$V_{-1}(2)|_{\mathbf{C_2}} = \mathbf{0.785398163}3859819 - 5.420140575251864 \cdot 10^{-15\checkmark}\, i$$

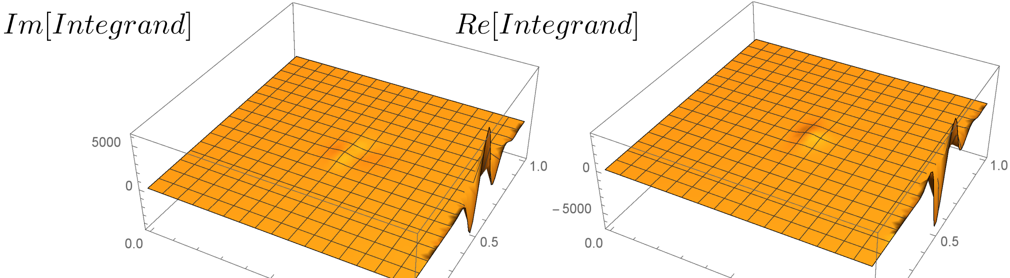
✓ $$V_{-1}(2)|_{\mathbf{C_3}} = \mathbf{0.785398163}2958756 + 2.435551760271437 \cdot 10^{-15\checkmark}\, i$$

Transformations of integration variables (Mappings)

$$\int_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty} dz_1 \int_{-\frac{2}{3}-i\infty}^{-\frac{2}{3}+i\infty} dz_2 \left(\frac{-s}{M_Z^2}\right)^{-z_1} \frac{\Gamma[-z_1]^3\Gamma[1+z_1]\Gamma[z_1-z_2]\Gamma[-z_2]^3\Gamma[1+z_2]\Gamma[1-z_1+z_2]}{s\ \ \Gamma[1-z_1]^2\Gamma[-z_1-z_2]\Gamma[1+z_1-z_2]}$$

Logarithmic (in MB.m, M. Czakon, CPC 2006):

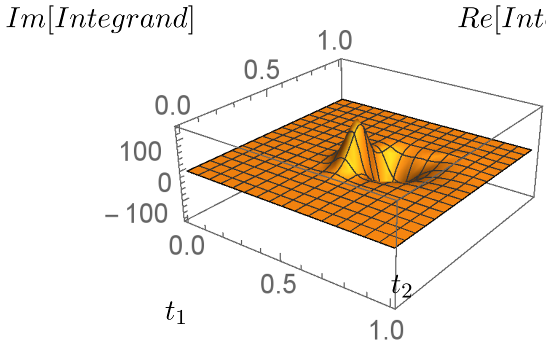
$$z_k \quad = \quad x_k + i\ln\left(\frac{t_k}{1-t_k}\right), \quad t_k \in (0,1), \quad \text{the Jacobians}: \quad J_k(t_k) = \frac{1}{t_k(1-t_k)}.$$

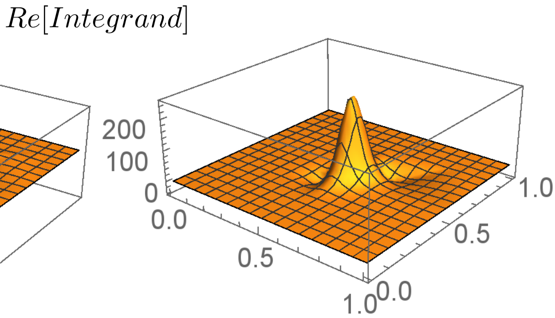


Transformations of variables (Mappings)

Tangent (in MBnumerics.m, ID, JU, TR, 2016):

$$z_k = x_k + i\,\frac{1}{\tan(-\pi t_k)},\quad t_k \in (0,1),\quad \text{the Jacobians}:\quad J_k = \frac{\pi}{\sin^2\left[(\pi t_k)\right]}.$$





The most difficult cases (for SD)