MB approach beyond two loops

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Outline

Starting point

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j \, x_j^{n_j-1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

The functions U and F are called graph or Symanzik polynomials.

Some remarks

Change of variables in Symanzik polynomials U and F is effective as:

- They are homogeneous in the Feynman parameters, U is of degree L, F is of degree $L+1$
- U is linear in each Feynman parameter. If all internal masses are zero, then also F is linear in each Feynman parameter
- In expanded form each monomial of U has coefficient $+1$

$$F = F_0 + U \sum_{i=1}^{N} x_i m_i^2$$

U and F from graph theory

C.Bogner, S.Weinzierl, arXiv:1002.3458

- Spanning tree T for the graph G
  sub-graph with the following properties:
  - T contains all the vertices of G
  - the number of loops in T is zero
  - T is connected

  T can be obtained from G by deleting L edges (L – number of loops in G)
- Spanning k–forest F for the graph G
  has the same properties as T but it is not required that a spanning forest is connected
  F can be obtained from G by deleting $L + k - 1$ edges

If $\mathcal{T}$ is the set of spanning forests of G and $\mathcal{T}_k$ is set of spanning k-forests of G when

$$\mathcal{T} = \bigcup_{k=1}^{r} \mathcal{T}_k \quad (r - \mathrm{numberofvertices})$$

($\mathcal{T}_k$ is the set of spanning trees)
Each element of $\mathcal{T}_k$ has k connected components $(T_1, \ldots, T_k)$
P_{T_i} is the set of external momenta attached to T_i for a given k–forest.

The spanning trees and the spanning 2–forests of a graph G are closely related to the graph polynomials U and F of the graph:

$$U = \sum_{T \in \mathcal{T}_1} \prod_{e_i \notin T} x_i$$

$$F = -\sum_{(T_1,T_2) \in \mathcal{T}_2} \left(\prod_{e_i \notin (T_1,T_2)} x_i \right) \left(\sum_{p_i \in P_{T_1}} p_i \right) \left(\sum_{p_j \in P_{T_2}} p_j \right) + U \sum_{i=1}^{n} x_i m_i^2 = F_0 + U \sum_{i=1}^{n} x_i m_i^2$$

Example:

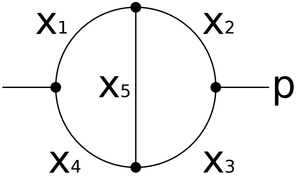


Figure 1: A two-loop two-point graph.

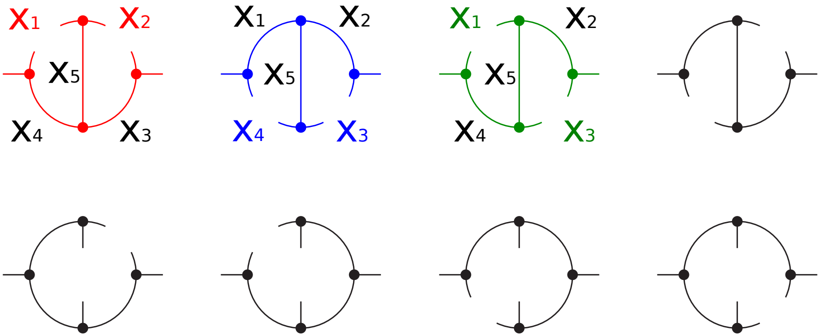


Figure 2: The set of spanning trees for the two-loop two-point graph of fig. 1.

$$U = x_1x_2 + x_3x_4 + x_1x_3 + x_2x_4 + x_1x_5 + x_2x_5 + x_3x_5 + x_4x_5$$

$$U = x_5(x_1 + x_4) + x_5(x_2 + x_3) + (x_1 + x_4)(x_2 + x_3)$$

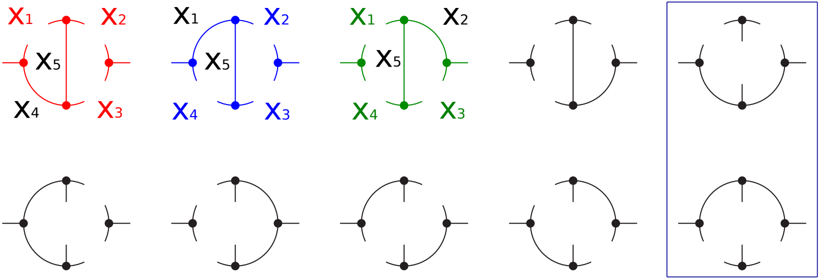
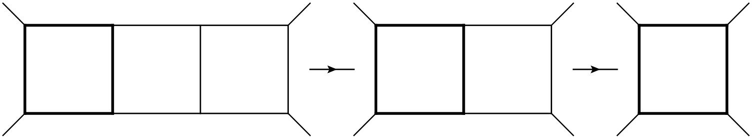


Figure 3: The set of spanning 2-forests for the two-loop two-point graph of fig. 1.

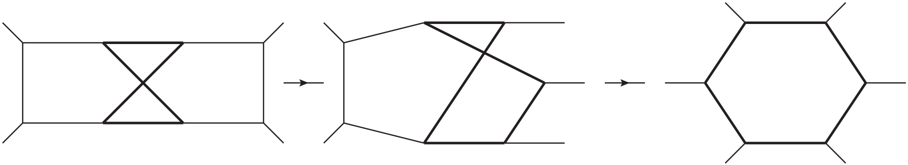
$$F = [x_1x_2x_3 + x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4 + x_2x_3x_5 + x_1x_4x_5 + x_2x_4x_5 + x_1x_3x_5]\,(-p^2)$$

Loop by loop (LA) approach

Planar case:



Non-planar case:



Global (GA) approach

U polynomial for non-planar 3-loop box (64 terms)

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x[1] x[2] x[4] + x[1] x[3] x[4] + x[2] x[3] x[4] + x[1] x[2] x[5] +
x[1] x[3] x[5] + x[2] x[3] x[5] + x[1] x[4] x[5] + x[2] x[4] x[5] +
x[2] x[4] x[6] + x[3] x[4] x[6] + x[2] x[5] x[6] + x[3] x[5] x[6] +
x[4] x[5] x[6] + x[2] x[4] x[7] + x[3] x[4] x[7] + x[2] x[5] x[7] +
x[3] x[5] x[7] + x[4] x[5] x[7] + x[1] x[2] x[8] + x[1] x[3] x[8] +
x[2] x[3] x[8] + x[1] x[4] x[8] + x[2] x[4] x[8] + x[2] x[6] x[8] +
x[3] x[6] x[8] + x[4] x[6] x[8] + x[2] x[7] x[8] + x[3] x[7] x[8] +
x[4] x[7] x[8] + x[1] x[2] x[9] + x[1] x[3] x[9] + x[2] x[3] x[9] +
x[2] x[4] x[9] + x[3] x[4] x[9] + x[1] x[5] x[9] + x[3] x[5] x[9] +
x[4] x[5] x[9] + x[2] x[6] x[9] + x[3] x[6] x[9] + x[5] x[6] x[9] +
x[2] x[7] x[9] + x[3] x[7] x[9] + x[5] x[7] x[9] + x[1] x[8] x[9] +
x[3] x[8] x[9] + x[4] x[8] x[9] + x[6] x[8] x[9] + x[7] x[8] x[9] +
x[1] x[2] x[10] + x[1] x[3] x[10] + x[2] x[3] x[10] +
x[1] x[4] x[10] + x[2] x[4] x[10] + x[2] x[6] x[10] +
x[3] x[6] x[10] + x[4] x[6] x[10] + x[2] x[7] x[10] +
x[3] x[7] x[10] + x[4] x[7] x[10] + x[1] x[9] x[10] +
x[3] x[9] x[10] + x[4] x[9] x[10] + x[6] x[9] x[10] + x[7] x[9] x[10]
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GA 2-loop example (non-planar vertex)

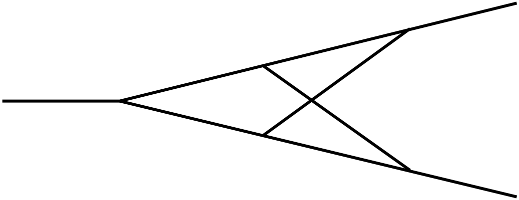


Figure 4: The non-planar vertex.

$$\iint d^dk_1 d^dk_2 \frac{1}{[k_1^2]^{n_1}[(p_1-k_1)^2]^{n_2}[(p_1-k_1-k_2)^2]^{n_3}} \frac{1}{[(p_2+k_1+k_2)^2]^{n_4}[(p_2+k_2)^2]^{n_5}[k_2^2]^{n_6}}$$

$$U = x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_3x_4 + x_1x_5 + x_2x_5 + x_4x_5 + x_2x_6 + x_3x_6 + x_4x_6 + x_5x_6$$

$$F = U\sum_{i=1}^{6} m_i^2 x_i - sx_1x_4x_5 - sx_1x_2x_6 - sx_1x_3x_6 - sx_2x_3x_6 - sx_1x_4x_6 - sx_1x_5x_6$$

Variables transformation

$$\{\vec{x}\}_i : x_k \to v_i \xi_{ik}$$

i denotes a subset of feynman parameters associated to propagators with different combinations of loop momenta

$$\begin{array}{rcll} m^2 = \sum x_i D_i & = & x_1 (p_1 - k_1 - k_2)^2 & \qquad x_1 \to v_1 \xi_{11} \\ & + & x_2 (p_2 + k_1 + k_2)^2 & \qquad x_2 \to v_1 \xi_{12} \\ & + & x_3 (k_1)^2 & \qquad x_3 \to v_2 \xi_{21} \\ & + & x_4 (p_1 - k_1)^2 & \qquad x_4 \to v_2 \xi_{22} \\ & + & x_5 (p_2 + k_2)^2 & \qquad x_5 \to v_3 \xi_{31} \\ & + & x_6 (k_2)^2 & \qquad x_6 \to v_3 \xi_{32} \end{array}$$

$$\delta\left(1 - \sum_{i=1}^{6} x_i\right) \Rightarrow \delta(1 - v_1 - v_2 - v_3)\delta(1 - \xi_{11} - \xi_{12})\delta(1 - \xi_{21} - \xi_{22})\delta(1 - \xi_{31} - \xi_{32})$$

Jacobian:

$$J = v_1^{N_{\xi_1} - 1} v_2^{N_{\xi_2} - 1} v_3^{N_{\xi_3} - 1} = v_1 v_2 v_3$$

- Using $\prod_i \delta\left(1-\sum_k \xi_{ik}\right)$ we can simplify U and F

$$U = v_1 v_2 + v_1 v_3 + v_2 v_3 \quad F = -s\xi_{11}\xi_{22}\xi_{31}v_1v_2v_3 - s\xi_{12}\xi_{21}\xi_{32}v_1v_2v_3$$
$$-s\xi_{31}\xi_{32}v_1v_3^2 - s\xi_{31}\xi_{32}v_2v_3^2$$

Chang–Wu theorem:
delta function in the feyman parameters representation can be replaced by

$$\delta\left(\sum_{i\in\Omega} x_i - 1\right)$$

where Ω is an arbitrary subset of the lines $1,\ldots,L$, when the integration over the rest of the variables, i.e. for $i \notin \Omega$, is extended to the integration from zero to infinity.

- Choose now v_3 as Chang-Wu variable $\int_0^\infty dv_3 \int_0^1 dv_1 dv_2 \delta(1-v_1-v_2)$

$$U = v_3 + v_1 v_2 \quad F = -s\xi_{11}\xi_{22}\xi_{31}v_1v_2v_3 - s\xi_{12}\xi_{21}\xi_{32}v_1v_2v_3$$
$$-s\xi_{31}\xi_{32}v_1v_3^2$$

- Apply MB relation for F

$$\frac{1}{(A_1+\ldots+A_n)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{(2\pi i)^{n-1}}\int_{-i\infty}^{i\infty} dz_1\ldots dz_{n-1}$$
$$\times \prod_{i=1}^{n-1} A_i^{z_i}\, A_n^{-\lambda-z_1-\ldots-z_{n-1}} \prod_{i=1}^{n-1}\Gamma(-z_i)\,\Gamma(\lambda+z_1+\ldots+z_{n-1})$$

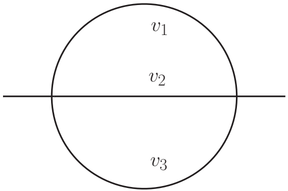
- Iintegrate over $v3$ using

$$\int_0^\infty dx\, x^{z_1}(x+y)^{z_2} = \frac{y^{1+z_1+z_2}\Gamma(1+z_1)\Gamma(-1-z_1-z_2)}{\Gamma(-z_2)}$$

- Iintegrate over each subset of variables $\{v,\xi_i\}$ separately using

$$\int_0^1 \prod_{i=1}^{N} dx_i\, x_i^{n_i-1}\,\delta(1-x_1-\ldots-x_N) = \frac{\Gamma(n_1)\ldots\Gamma(n_N)}{\Gamma\,(n_1+\ldots+n_N)}$$

U polynomial gives no additional MB integration and final dimentionality depends only on length of F $\longrightarrow$ similar to one loop integrals and/or LA approach



$$U = v_1 v_2 + v_1 v_3 + v_2 v_3$$
$$F = -p^2 v_1 v_2 v_3 + U \sum_i v_i m_i^2$$

$$G(X) \sim \int \prod d\xi_{ik} \delta \left(1 - \sum_k \xi_{ik} \right)$$
$$\int d^d k_1 d^d k_2 \frac{1}{[k_1^2 - m_1^2(S,\xi_{ik})]^{n_1} [k_2^2 - m_2^2(S,\xi_{ik})]^{n_2} [(p + k_1 + k_2)^2 - m_3^2(S,\xi_{ik})]^{n_3}}$$
$$p^2 = Q(S,\xi_{ik})$$

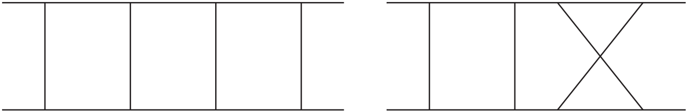
In case of massless non-planar vertex from above

$$p^2 = -s(\xi_{12}\xi_{22}\xi_{31} + \xi_{12}\xi_{21}\xi_{32} - \xi_{31}\xi_{32})$$

$$m_1^2 = -s\xi_{31}\xi_{32}, \quad m_2^2 = m_3^2 = 0.$$

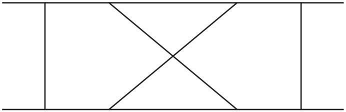
3-loop GA

Case I:



$$U = v_1 v_2 v_3 + v_1 v_2 v_4 + v_2 v_3 v_4 + v_1 v_2 v_5 + v_1 v_3 v_5 + v_2 v_3 v_5 + v_1 v_4 v_5 + v_3 v_4 v_5$$

Case II:



$$U = v_1 v_2 v_3 + v_1 v_2 v_4 + v_1 v_3 v_4 + v_1 v_2 v_5 + v_1 v_3 v_5 + v_2 v_3 v_5 + v_2 v_4 v_5 + v_3 v_4 v_5 \\ + v_1 v_2 v_6 + v_2 v_3 v_6 + v_1 v_4 v_6 + v_2 v_4 v_6 + v_3 v_4 v_6 + v_1 v_5 v_6 + v_3 v_5 v_6 + v_4 v_5 v_6$$

Now in the Chang-Wu theorem we choose 3 variables

$$\int_0^\infty dv_2 dv_3 dv_4 \int_0^1 dv_1 dv_5 dv_6 \delta(1 - v_1 - v_5 - v_6)$$

$$U_{CW} = v_2 v_3 + v_2 v_4 + v_3 v_4 + v_1 v_2 v_5 + v_1 v_3 v_5 + v_1 v_2 v_6 + v_1 v_4 v_6 + v_1 v_5 v_6 + v_3 v_5 v_6 + v_4 v_5 v_6$$

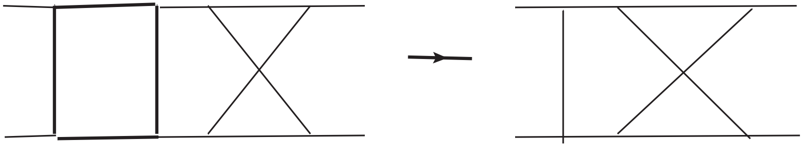
Factorization trick:

$$U_{CW} = v_2(v_3 + v_4 + v_1 v_5) + v_3(v_4 + v_1 v_5) + v_1 v_6(v_2 + v_5) + v_4 v_6(v_1 + v_5) + v_3 v_5 v_6$$

U polynomial gives 4 additional MB integration!
GA usually gives optimal representation if from the beginning Lenght(U) $\sim$ Lenght(F)

3-loop mixed approach

Mixed approach starting with planar subloop:



Mixed approach starting with non-planar subloop: