Numerical evaluation of Mellin-Barnes integrals in Minkowskian kinematics Workshop Katowice

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Outline

Asymmetries measured at the Z pole

We study the process $e^+e^- \to (Z) \to b\bar{b}$

Pseudo-observables, unfolded at the Z peak

forward-backward asymmetry $A_{\mathrm{FB}}^{\mathrm{b\bar{b}},0} = \frac{3}{4} A_{\mathrm{e}} A_{\mathrm{b}}$

f-b left-right asymmetry $A_{\mathrm{FB,LR}}^{\mathrm{b\bar{b}},0} = \frac{3}{4} P_{\mathrm{e}} A_{\mathrm{b}}$, P_{e} is the electron polarization

$$A_{\mathrm{b}} = \frac{2\Re e \frac{v_b}{a_b}}{1 + \left(\Re e \frac{v_b}{a_b}\right)^2} = \frac{1 - 4|Q_b| \sin^2\theta_{\mathrm{eff}}^{\mathrm{b}}}{1 - 4|Q_b| \sin^2\theta_{\mathrm{eff}}^{\mathrm{b}} + 8Q_b^2 (\sin^2\theta_{\mathrm{eff}}^{\mathrm{b}})^2} \quad (1)$$

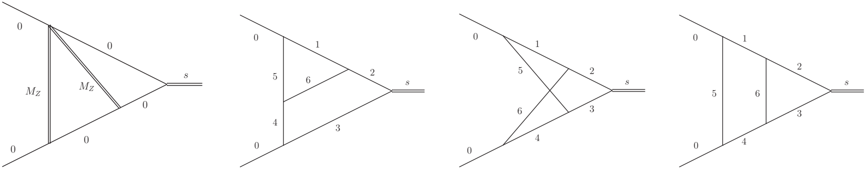
Definition of the effective weak mixing angle

$$\sin^2\theta_{\mathrm{eff}}^{\mathrm{b}} = \frac{1}{4|Q_b|} \left(1 - \Re e \frac{v_b}{a_b}\right) \quad (2)$$

v_b and a_b are effective vector coupling and axial-vector coupling of the $Zb\bar{b}$ vertex

Samples of Feynman integral topologies for the $Z\bar{b}b$ vertex

After projection only scalar integrals remain, but may contain non-trivial combinations of scalar products in the numerator.



(a) (b) (c) (d)

Some non trivial cases:

- (b) $m_4 = \{M_Z\}$ and $m_1 = \{M_W, m_t\}$ and $m_5 = m_6 = \{m_t, M_W\}$
- (c) $m_1 = m_4 = \{M_Z\}$
- (d) $m_5 = \{M_Z\}$ and $m_6 = \{M_W, m_t\}$ and $m_2 = m_3 = \{m_t, M_W\}$

Two independent numerical calculations

- The integrals contain up to three dimensionless parameters

$$\left\{ \frac{M_H^2}{M_Z^2}, \frac{M_W^2}{M_Z^2}, \frac{m_t^2}{M_Z^2}, \frac{(M_Z^2 + i\varepsilon)}{M_Z^2} \right\} \tag{3}$$

- Many of them contain ultraviolet and infrared singularities, even though the divergences cancel in the final result
- In general, it is not possible to compute all integrals analytically with available methods and tools, but instead one has to resort to numerical integration strategies
- The aim is to obtain eight significant digits, to be obtained with two completely independent numerical calculations

Numerical Methods

Sector decomposition

FIESTA 3 [A.V.Smirnov, 2014] and SecDec 3 [Borowka, et. al., 2015]

Mellin-Barnes integral approach

- With AMBRE 2 [Gluza, et. al., 2011] (AMBRE 3 [Dubovyk, et. al., 2015]) we derived Mellin-Barnes representations for planar (non-planar) topologies. One may use PlanarityTest [Bielas, et. al, 2013] for automatic identification.
- Expansion in terms of $\epsilon = (4 - D)/2$ is done with MB [Czakon, 2006], MBresolve [A. Smirnov, V. Smirnov, 2009], barnesroutines (D. Kosower).
- For the numerical treatment of massive Mellin-Barnes integrals with Minkowskian kinematics, the package MBnumerics [Dubovyk, Riemann, Usovitsch] is being developed since 2015.

Mellin-Barnes integral construction

Master formula: $$\frac{1}{(A+B)^{\nu}} = \frac{B^{-\nu}}{2\pi i\Gamma[\nu]} \int_{-i\infty}^{+i\infty} \mathrm{d}\sigma A^{\sigma} B^{-\sigma} \Gamma[-\sigma]\Gamma[\nu+\sigma] \quad (4)$$

Loop integral with loop momenta k and propagator internal momenta q:

$$G_L[\not{T}(k)] = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \ldots d^d k_L \quad \not{T}(k)}{(q_1^2 - m_1^2)^{\nu_1} \ldots (q_i^2 - m_i^2)^{\nu_j} \ldots (q_N^2 - m_N^2)^{\nu_N}} \quad (5)$$

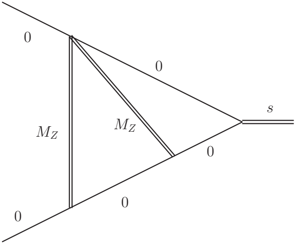
Momentum integrals are replaced by the Feynman parameter integrals:

$$= \frac{(-1)^{N_\nu} \Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^{N} \Gamma(\nu_i)} \int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\nu_j - 1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}} \quad (6)$$

Introduce the master formula and eliminate all x integration variables with:

$$\int_0^1 \prod_{i=1}^{N} dx_i \; x_i^{q_i - 1} \; \delta\left(1 - \sum_j x_j\right) = \frac{\Gamma(q_1) \cdots \Gamma(q_N)}{\Gamma\left(q_1 + \cdots + q_N\right)} \quad (7)$$

Mellin-Barnes Coefficients in the ϵ expansion, example 1



- Known analytic result for Master integrals in [Aglietti, Bonciani,2004]
- Mellin-Barnes integral coefficient in the ϵ expansion:

$$\mathcal{I}(z_1, z_2) = \frac{-(-\frac{s}{M_Z^2})^{1+z_1}\Gamma[-z_1]\Gamma[1+z_1]^2\Gamma[1+z_1-z_2]\Gamma[-z_2]\Gamma[1+z_2]^3\Gamma[-z_1+z_2]}{2\Gamma[3+z_1]\Gamma[1-z_1+z_2]\Gamma[2+z_1+z_2]} \quad (8)$$

- Mellin-Barnes integration variables $z_i = x_i + i\,t_i$, where the x_i are fixed and $t_i \in (-\infty, +\infty)$
- Here $x_1 = -\frac{2}{3}$, $x_2 = -\frac{1}{3}$

Numerical integration in MB.m and MBnumerics.m

$$z_i^{\mathrm{L}} = x_i + i \ln\left(\frac{t_i}{1-t_i}\right), \; t_i \in (0,1), \text{ Jacobians: } J_i^{\mathrm{L}} = \frac{i}{t_i(1-t_i)} \quad (9)$$

$$I^{\mathrm{MB.m}} = \frac{1}{(2\pi i)^2} \int_0^1 \int_0^1 J_1^{\mathrm{L}} J_2^{\mathrm{L}} \mathcal{I}\left(z_1^{\mathrm{L}}, z_2^{\mathrm{L}}\right) \mathrm{d}t_1 \mathrm{d}t_2 \quad (10)$$

$$z_i^{\mathrm{T}} = x_i + n_i + \frac{(i+\theta)}{\tan(-\pi\, t_i)}, \; t_i \in (0,1), \text{ Jacobians: } J_i^{\mathrm{T}} = \frac{(i+\theta)\pi}{\sin^2(\pi\, t_i)} \quad (11)$$

$$I^{\mathrm{MBnumerics.m}}(n_1, n_2) = \frac{1}{(2\pi i)^2} \int_0^1 \int_0^1 J_1^{\mathrm{T}} J_2^{\mathrm{T}} \hat{\mathcal{I}}\left(z_1^{\mathrm{T}}, z_2^{\mathrm{T}}\right) \mathrm{d}t_1 \mathrm{d}t_2 \quad (12)$$

- $n_i \in \mathrm{Integers}$, shifts [Anastasiou, Daleo, 2006]
- $\theta \in \mathrm{Reals}$, rotations [Freitas, Huang, 2010]
- tangent mapping imposes $\mathcal{I}[\prod_i \Gamma_i] \to \hat{\mathcal{I}}[e^{\sum_i \ln(\Gamma_i)}]$
- $I^{\mathrm{MBnumerics.m}}(n_1, n_2)$ is now a discrete function in n_i

Asymptotic behavior in generalized spherical coordinates for $r \to \infty$

Euclidean kinematics

$$\Re e\left(\lim_{r\to\infty} \mathcal{I}\right) \approx \frac{e^{-\beta\, r}}{r^{\alpha}},\ \beta > 0 \text{ and } \alpha \text{ arbitrary} \tag{13}$$

for any angular direction

Minkowskian kinematics (physical momenta)

$$\Re e\left(\lim_{r\to\infty} \mathcal{I}\right) \approx \frac{1}{r^{\alpha}},\ \alpha \text{ arbitrary} \tag{14}$$

- logarithmic mapping always has an infinity at the boundary
- For $\alpha \geq 2$ tangent mapping has no infinities at the boundaries
- $\alpha < 2$ in either mapping the integrand is not absolutely convergent

Rotations and Shifts

Contour rotations θ. The transformations $z_i = x_i + (i + \theta)\, t_i$ do not cross poles and may introduce $\beta > 0$ for any angular direction in Minkowskian kinematics.

Contour shifts n_i. Treat the Mellin-Barnes integrals as discrete functions: $z_i = x_i + n_i + i\, t_i$.

- May improve convergence by shifting: $\Re e(\lim_{r\to\infty} \mathcal{I}) \approx \frac{1}{r^{\alpha(\{n_i + x_i\})}}$ $\alpha \geq 2$.
- Add up all crossed poles (integrals with one dimension less)
- May reduce the order of magnitude of the shifted integral
- The shifted integral and its poor knowledge becomes numerically less important
- In effect, the procedure consists of a summing over a finite number of residues with a controlled remainder.

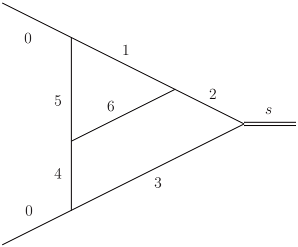
Numerical effects

Point in the kinematics: $s = 3 + i10^{-16}$, $M_Z^2 = 1$

$$
\begin{aligned}
I^{\text{MB.m}} &= 0.0696190689628302 + 1.705511849228807\, i \\
I^{\text{tangent}} &= 0.0696190691506288 + 1.705511853846761\, i \\
I^{\text{logarithm}}_{\text{rotation}} &= 0.0696190691545005 + 1.705511853839675\, i \\
I^{\text{tangent}}_{\text{rotation}} &= 0.0696190691545014 + 1.705511853839673\, i \\
I^{\text{MBnumerics.m}} &= 0.0696190691545014 + 1.705511853839671\, i
\end{aligned}
\tag{15}
$$

- Best optimization in MB.m is used
- 500 000 evaluation points are used
- All calculations are evaluated with CUHRE (CUBA) [Hahn, 2015]
- Since $\frac{1}{t^\alpha}$, $\alpha > 2$, tangent mapping gives improvement
- After contour rotation the mapping is not mandatory
- MBnumerics.m takes control over shifts, rotations, mappings

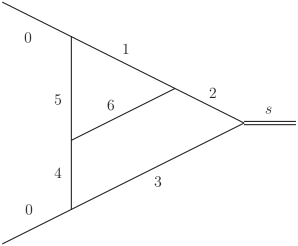
Mellin-Barnes Coefficients in the ϵ expansion, example 2



- $m_1 = m_t$ and $m_5 = m_6 = M_W$

$$\begin{aligned}
\text{SD} &= 1.54223108 + 0.24726796i \\
&+\frac{1}{\epsilon}(0.1232504106 - 1.0618605150i) \\
&-\frac{1}{\epsilon^2}(0.3380005111 + 8.5 * 10^{(-10)}i) \\
\text{MB} &= 1.542452628 5 + 0.2473136625i \\
&+\frac{1}{\epsilon}(0.12324963277 - 1.06185992255i) \\
&+\frac{1}{\epsilon^2}(-0.3380005111031245 + 0i)
\end{aligned}$$

Mellin-Barnes Coefficients in the ϵ expansion, example 3

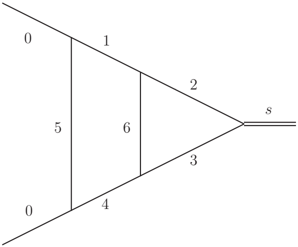


- $m_1 = m_t$ and $m_5 = m_6 = M_W$ and $m_4 = M_Z$

SD $= 0.1758981641 + 0.7088901117i$

MB $= 0.1758981962 + 0.7088900264i$

Mellin-Barnes Coefficients in the ϵ expansion, example 4



- $m_6 = M_W$

$$\text{SD} = 16.03548\textcolor{red}{5}9574734 \quad + 22.85602\textcolor{red}{8}518577i$$

$$\text{MB} = 16.035396066\textcolor{red}{}5095 \quad + 22.85597198\textcolor{red}{2}202i$$

$$\text{HPL} = 16.03539606678437 + 22.855971982557442i$$

- MB error is coming from the application of the shifts and the Monte Carlo integration techniques
- SD erros is only from the Monte Carlo integration techniques

Conclusions and Outlook

- We calculate the asymmetry parameter A_{b} which can be related to the asymmetric pseudo-observables
- The main challenge was the calculation of massive two-loop vertex diagrams
- No reduction of integrals to masters
- New automatized tools AMBRE 3 and MBnumerics for the evaluation of the Mellin-Barnes integrals in Minkowskian kinematics together with sector decomposition programs SecDec 3 and Fiesta 3