1. Stationary phase contour/manifold

$$I(s,t,\ldots) = \int_{i\mathbb{R}\times\ldots\times i\mathbb{R}} dz_1 \ldots dz_n \, F(z) = \int_{i\mathbb{R}\times\ldots\times i\mathbb{R}} dz_1 \ldots dz_n \, e^{-[\mathrm{Re} f(z) + i \mathrm{Im} f(z)]}$$

Integrate over a n-dimensional manifold $\mathcal{M}$ along which $\mathrm{Im} f = \mathrm{const}$:

$$I(s,t,\ldots) = e^{-i\mathrm{Im} f} \int_{\mathcal{M}} dt_1 \ldots dt_n \, \mathrm{Jac}(t,z) \, e^{-\mathrm{Re}\, f}$$

$\mathcal{M}$ is parametrized by $t_1, t_2, \ldots, t_n$.

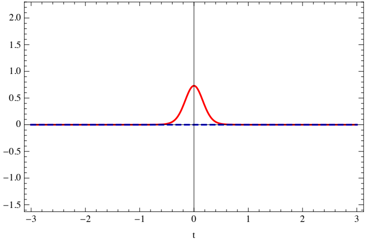
2. Steepest descent contour $\mathcal{C}$

$$\mathcal{C}(z_*) = \{z(\tau) \in \mathbb{C}^n : \frac{dz_i}{d\tau} = (\partial_{z_i} f)^*, \quad z(0) = z_*, \quad \partial_{z_i} f(z_*) = 0\}$$

Along $\mathcal{C}(z_*)$:

$$\mathrm{Im}\, f = \text{const}, \qquad \frac{d}{d\tau} \mathrm{Re}\, f < 0$$

E.g. for $F(z) = (-s)^{-z} \frac{\Gamma^3(-z)\Gamma(z+1)}{\Gamma(-2z)}$ and $s = -1/20$:



3. Lefschetz thimble

Natural candidate for $\mathcal{M}$, so-called Lefschetz thimble $\mathcal{J}(z_*)$:

$$\mathcal{J}(z_*) = \text{a union of all steepest descent contours } \mathcal{C}(z_*)$$
$$\dim_{\mathbb{R}} \mathcal{J}(z_*) = n$$

But $\frac{dz_i}{d\tau} = (\partial_{z_i} f)^*$ gives dependence only on one coordinate.
Dependence on $t_2, \ldots, t_n$ is not known!

In other words, one needs aditional $n-1$ operators $\mathcal{O}_k(f)$ which restrict hypersurface $\operatorname{Im} f = \text{const}$ to $\mathcal{M}$:

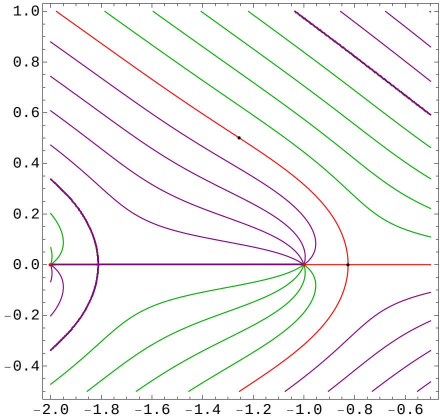
$$\operatorname{Im} f = \text{const}, \quad \mathcal{O}_k(f) =? \quad (k = 1, \ldots, n-1)$$

4. 1-dimensional case

$\operatorname{Im} f = \text{const}$ is enough to find $\mathcal{J}(z_*)$ in $n = 1$ case:

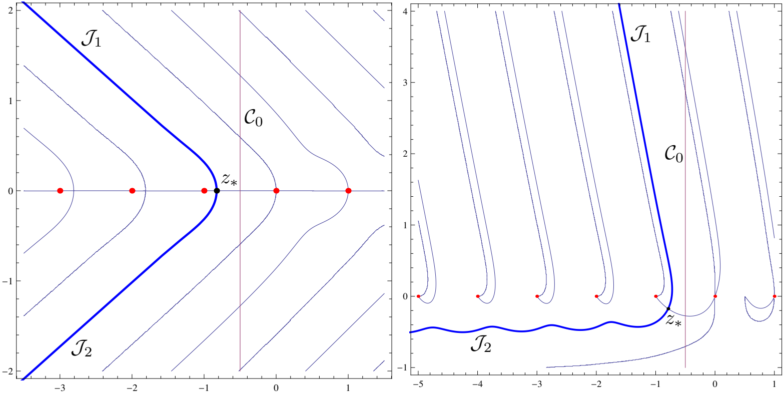
$$\dim_{\mathbb{R}}\{z \in \mathbb{C} \,|\, \operatorname{Im} f = \text{const}\} = 1 = \dim_{\mathbb{R}} \mathcal{J}(z_*)$$

E.g. for $F(z) = (-s)^{-z}\frac{\Gamma^3(-z)\Gamma(z+1)}{\Gamma(-2z)}$ and $s = -1/20$:



5. 1-dimensional case

$$F(z) = (-s)^{-z}\,\frac{\Gamma^3(-z)\Gamma(z+1)}{\Gamma(-2z)} \text{ for } s = -1/20 \text{ and } s = 1 + i0^+:$$



6. 1-dimensional case

$$I(s) = \sum_{k=1,2} \frac{e^{-i\,\mathrm{Im}\, f|_{\mathcal{J}_k}}}{2\pi i} \int_{\mathcal{J}_k} dz\, e^{-\mathrm{Re}\, f} + \frac{1}{2\pi i}\int_{\mathcal{A}} dz\, F + \sum_{\mathcal{C}_0 \to \mathcal{C}} \mathrm{Res}\, F$$