Singular values, contractions, dilations and neutrino mixing analysis*

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In collaboration with:
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* Dedicated to Marek Zrałek.

M and U

Different neutrino mass models, e.g. type-I,II,III seesaw models, radiative, etc. can be embedded into general M-form:

$$M = \begin{pmatrix} 0\{m_L\} & m_D \\ m_D^T & m_R \end{pmatrix}, \quad M_{diag} = U^T M U, \quad U = \begin{pmatrix} V_{ll} & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix}$$

- We know much about M_{diag}, V_{ll} (experiments), also about V_{lh} (indirectly), and V_{hh} (model dependent)
- Certainly, they are interconnected, how much can matrix theory help formally and in practical way to dig in U and M?

$\longrightarrow$ *Poster by Wojtek Flieger on mass spectrum.*

In this talk: Focus on U

K. Bielas, W. Flieger, JG, M. Gluza
"A novel approach to neutrino mixing analysis based on singular values", arXiv:1708.09196

Useful Appendices, e.g.:

Theorem

Krein-Milman
Let X be a topological vector space in which the dual space X^* separates points. If A is compact, convex set in X, then A is closed, convex hull of its extreme points.

Proposition

Once a set of matrix contractions is given, the convex hull with vertices at this set contains only contractions.

Proof in 1708.09196, etc.

Matrix theory and convex geometry

- R. A. Horn and C. R. Johnson,
  Matrix analysis (Cambridge U. Press, 2012).
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In physics (and theory):

Singular values ($\sim$ known), **contractions**, **dilations** (poorly known)

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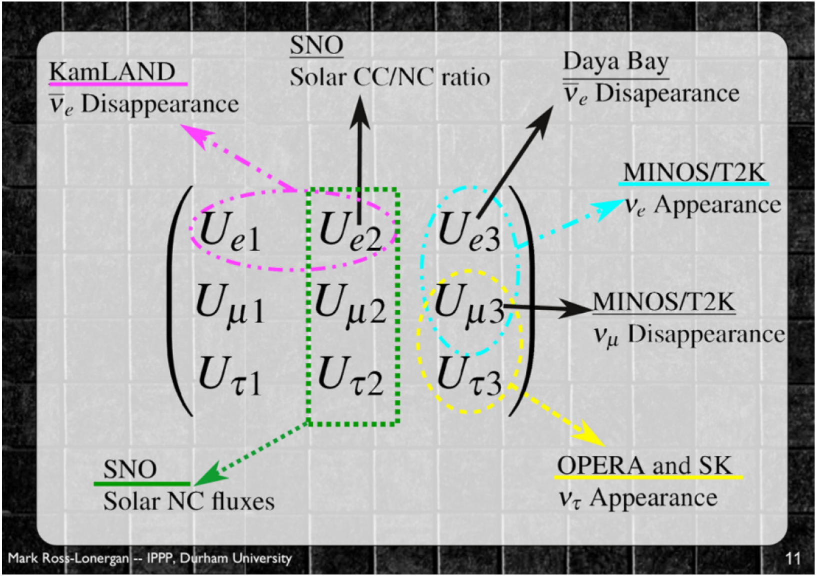
We try to bridge mathematical knowledge to neutrino mixing

What is measured?



Neutrino mixing in the Standard Model

$$\nu_\alpha^{(f)} = (U_{PMNS})_{\alpha i} \nu_i^{(m)}$$

Mixing matrix

$$U_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Experimental values of mixing parameters

$$\theta_{12} \in [31.38^\circ, 35.99^\circ], \quad \theta_{23} \in [38.4^\circ, 53.0^\circ],$$
$$\theta_{13} \in [7.99^\circ, 8.91^\circ], \quad \delta \in [0, 2\pi]$$

Full experimental data - interval matrix

$$U_{PMNS} \xrightarrow{\theta_{1,2,3},\delta} V_{osc}$$

CP Invariant Case

$$\mathrm{V_{osc}} = \begin{pmatrix} 0.799 \div 0.845 & 0.514 \div 0.582 & 0.139 \div 0.155 \\ -0.538 \div -0.408 & 0.414 \div 0.624 & 0.615 \div 0.791 \\ 0.22 \div 0.402 & -0.73 \div -0.567 & 0.595 \div 0.776 \end{pmatrix}$$

$$V_{osc} \xrightarrow{?} BSM$$

No much clues for that (usually bounds on masses and couplings):

1. 20 million *Z*-**boson decays**, yielding $N_\nu = 2.9840 \pm 0.0082$
2. Liquid Scintillator Neutrino Detector (**LSND**) experiment an excess of $\bar{\nu}_e$ appearing in a mostly $\bar{\nu}_\mu$ beam at 3.8σ-level $\longrightarrow$ **sterile**

Extended mixing - BSM models

Complete mixing

$$\begin{pmatrix} \nu^{(f)} \\ \hat{\nu}^{(f)} \end{pmatrix} = \begin{pmatrix} V_{osc} & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} \begin{pmatrix} \nu^{(m)} \\ \hat{\nu}^{(m)} \end{pmatrix} \equiv U \begin{pmatrix} \nu^{(m)} \\ \hat{\nu}^{(m)} \end{pmatrix}$$

Observable part

$$\nu_\alpha^{(f)} = \underbrace{(V_{osc})_{\alpha i} \nu_i^{(m)}}_{\text{SM part}} + \underbrace{(V_{lh})_{\alpha j} \hat{\nu}_j^{(m)}}_{\text{BSM part}}$$

A standard approach to deviation from unitarity

$$\not{U}_{PMNS} \not{U}^\dagger_{PMNS} \equiv [(1+\eta)N][(1+\eta)N]^\dagger = 1 + \epsilon$$

$$N - \text{Unitary}$$

$$\eta, \epsilon - \text{Hermitian}$$

Our approach: mixing matrix and singular values

Singular values σ_i of a given matrix A are positive square roots of the eigenvalues λ_i of the matrix $AA^\dagger$

$$\sigma_i(A) = \sqrt{\lambda_i(AA^\dagger)}$$

Properties:

- generalization of eigenvalues
- always positive
- stable under small perturbations (controlling error estimation)

Unitary matrices

$$UU^\dagger = I = diag(1, 1, ..., 1) \Longrightarrow \text{all singular values equal to 1}$$

Characterization of physical mixing matrices

$$\begin{pmatrix} V_{osc} & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} \quad ?$$

$\Downarrow$

Contraction

$$\| V_{osc} \| \leq 1$$

Contractions

$$\|A\| \leq 1$$

Operator norm (spectral norm)

$$\|A\| := \sup_{\|x\|=1} \|Ax\| = \sigma_{\max}(A)$$

Contractions as submatrices of the unitary matrix

$$\left\| \begin{pmatrix} V_{osc} & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} \right\| = 1 \Longrightarrow \|V_{osc}\| \leq 1$$

Unitarity and Contraction: a toy example

For U_{PMNS} holds

$$\sum_{\alpha} P_{\alpha\beta} = 1,$$

However, for a nonunitary U this relation is not fulfilled. $\Theta_2 = \Theta_1 + \epsilon$

$$U = \begin{pmatrix} \cos\Theta_1 & \sin\Theta_1 \\ -\sin\Theta_2 & \cos\Theta_2 \end{pmatrix}$$

In this case we get, $\Delta_{ij} \propto (m_i^2 - m_j^2)\frac{L}{E}$

$$P_{ee} + P_{e\mu} = 1 + 4\epsilon \sin^2\Delta_{21} \sin\Theta_1 \cos\Theta_1 \cos 2\Theta_1 + \mathcal{O}(\epsilon^2)$$

$$P_{\mu e} + P_{\mu\mu} = 1 - 4\epsilon \sin^2\Delta_{21} \sin\Theta_1 \cos\Theta_1 \cos 2\Theta_1 + \mathcal{O}(\epsilon^2)$$

Peculiar fact:

$$\|U\| \geq 1$$

Non-physical parametrization!

Statistics of Contractions in V_{osc}

Experimental mixing matrix

$$V_{\rm osc} = \begin{pmatrix} 0.799 \div 0.845 & 0.514 \div 0.582 & 0.139 \div 0.155 \\ -0.538 \div -0.408 & 0.414 \div 0.624 & 0.615 \div 0.791 \\ 0.22 \div 0.402 & -0.73 \div -0.567 & 0.595 \div 0.776 \end{pmatrix}$$

Contractions: only 4 %

Non-physical: 96%!

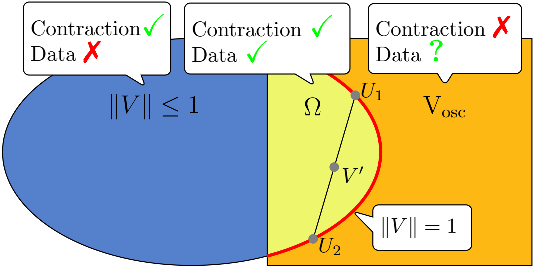
Contractions as a convex combination of unitary matrices

$$V = \sum_{i=1}^{m} \alpha_i U_i, \quad \alpha_i \geq 0 \text{ and } \sum_{\rm i=1}^{\rm m} \alpha_1 = 1$$

$$\|V\| = \|\sum_{i=1}^{m} \alpha_i U_i\| \leq \sum_{i=1}^{m} \alpha_i \|U_i\| = 1$$

Physical Region

$$\Omega := \mathrm{conv}(U_{PMNS}) = \{\sum_{i=1}^{m} \alpha_i U_i \mid U_i \in U(3), \alpha_1, ..., \alpha_m \geq 0, \sum_{i=1}^{m} \alpha_i = 1,$$
$$\theta_{12}, \theta_{13}, \theta_{23} \text{ and } \delta \text{ given by experimental values}\}$$



Unitary dilation

Contractions

$$U_{PMNS} \to V_{osc} \xrightarrow{\text{contractions}} \Omega$$

BSM?

$$V \in \Omega \xrightarrow{\text{dilation}} \begin{pmatrix} V & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} \equiv U \to UU^\dagger = I$$

CS decomposition

$$U \equiv \begin{pmatrix} V & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \left(\begin{array}{c|cc} C & -S & 0 \\ \hline S & C & 0 \\ 0 & 0 & I_{m-n} \end{array}\right) \begin{pmatrix} Q_1^\dagger & 0 \\ 0 & Q_2^\dagger \end{pmatrix}$$

where $C \geq 0$ and $S \geq 0$ are diagonal matrices satisfying $C^2 + S^2 = I_n$
$W_1, Q_1 \in M_{n\times n}$ and $W_2, Q_2 \in M_{m\times m}$ are unitary matrices

Unitary dilation: an example

As an illustration let us take two U_{PMNS} matrices

$$U_1 : \ \theta_{12} = 31.38^\circ, \theta_{23} = 38.4^\circ, \theta_{13} = 7.99^\circ,$$
$$U_2 : \ \theta_{12} = 35.99^\circ, \theta_{23} = 52.8^\circ, \theta_{13} = 8.90^\circ,$$

and let us construct a contraction as

$$V' = \frac{1}{2} U_1 + \frac{1}{2} U_2,$$

The set of singular values

$$\sigma_1(V') = 1, \ \boldsymbol{\sigma_2(V') = 0.991}, \ \boldsymbol{\sigma_3(V') = 0.991}$$

for which we get the following unitary dilation

$$U = \left(\begin{array}{ccc|cc} 0.822411 & 0.548133 & 0.146854 & 0.0169583 & -0.0368511 \\ -0.468394 & 0.520442 & 0.70103 & -0.133845 & 0.0197681 \\ 0.311417 & -0.643236 & 0.686702 & 0.0250273 & 0.130689 \\ \hline -0.0524981 & 0.122242 & -0.0336064 & 0.599485 & 0.788536 \\ -0.0671638 & 0.00403263 & 0.119588 & 0.788536 & -0.599485 \end{array}\right)$$

Quark Sector

Wolfenstein parametrization

$$s_{12} = \lambda, \quad s_{23} = A\lambda^2, \quad s_{13}e^{i\delta} = A\lambda^3(\rho + i\eta)$$

$$V_{CKM} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(\rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

Distribution of contractions

All matrices within V_{CKM} are contractions with 2‰ accuracy

$$6\% \text{ of } \|\mathrm{V_{CKM}}\| = 1.002, \qquad 0.961 \le \|\mathrm{V_{osc}}\| \le 1.178$$

$$94\% \text{ of } \|\mathrm{V_{CKM}}\| = 1.001 \qquad \sim 4\ \% \qquad \sim 18\ \%$$

Summary

- Interval matrix V_{osc} allows for independent analysis of mixing data
- **Matrix theory** and **convex geometry** offer suitable tools for that
- **Singular values** enrich studies beyond unitarity
- **Contractions** are natural to describe interplay between SM and BSM mixing theories in V_{osc}. They define physical region Ω by U_{PMNS} **convex** combination.
- There is a lot of space for BSM in V_{osc}.
  Dilations allow for appropriate construction of complete unitary matrices

Outlook: Advanced theory for M and $M \longleftrightarrow U$.

Backup slides

Matrix norm

A matrix norm is a function $\|\cdot\|$ from the set of all complex (real matrices) into $\mathbb{R}$ that satisfies the following properties

$$\|A\| \geq 0 \text{ and } \|\mathrm{A}\| = 0 \iff \mathrm{A} = 0,$$
$$\|\alpha A\| = |\alpha|\|A\|, \alpha \in C,$$
$$\|A + B\| \leq \|A\| + \|B\|,$$
$$\|AB\| \leq \|A\|\|B\|$$

Examples of matrix norms

- spectral norm: $\|A\| = \max_{\|x\|_2=1} \|Ax\|_2 = \sigma_1(A)$
- Frobenius norm: $\|A\|_F = \sqrt{Tr(A^\dagger A)} = \sqrt{\sum_{i,j=1}^n |a_{ij}|^2} = \sqrt{\sum_{i=1}^n \sigma_i^2}$
- maximum absolute column sum norm: $\|A\|_1 = \max_{\|x\|_1=1} \|Ax\|_\infty = \max_j \sum_i |a_{ij}|$
- maximum absolute row sum norm: $\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_i \sum_j |a_{ij}|$

Weyl's inequality for singular values

Let A and B be a $m \times n$ matrices and let $q = \min\{m, n\}$. Then

$$\sigma_j(A+B) \leq \sigma_i(A) + \sigma_{j-i+1}(B) \text{ for } i \leq j$$

Proof of the toy example

Let us calculate UU^T and U^TU for U, $s(c)_i \equiv \sin(\cos)\Theta_i,\ i = 1, 2$

$$UU^T = \begin{pmatrix} 1 & s_1c_2 - s_2c_1 \\ s_1c_2 - s_2c_1 & 1 \end{pmatrix}$$

$$U^TU = \begin{pmatrix} c_1^2 + s_2^2 & c_1s_1 - s_2c_2 \\ c_1s_1 - s_2c_2 & s_1^2 + c_2^2 \end{pmatrix}$$

As for the real A we have $\|A^TA\| = \|AA^T\| = \|A\|^2$, we can focus only on one of this products. Let us then write UU^T in the following form

$$UU^T = \begin{pmatrix} 1 & s_1c_2 - s_2c_1 \\ s_1c_2 - s_2c_1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & s_1c_2 - s_2c_1 \\ s_1c_2 - s_2c_1 & 0 \end{pmatrix}$$

Proof of the toy example

This can be simplified into

$$UU^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & s_3 \\ s_3 & 0 \end{pmatrix} \equiv I + B$$

where $s_3 \equiv \sin\Theta_3 = \sin(\Theta_1 - \Theta_2)$.
Let us observe that eigenvalues of B are equal $\pm s_3$.
Using fact that spectral norm is unitarily invariant and matrix B is symmetric, we get

$$\begin{aligned} \|UU^T\| &= \|I + B\| = \|W^T(I + B)W\| = \|I + W^T BW\| \\ &= \|I + D\| \end{aligned}$$

where W is an orthogonal matrix such that

$$W^T BW = D = diag(s_3, -s_3)$$

Proof of the toy example

Since $I + D$ equals

$$\begin{pmatrix} 1 + s_3 & 0 \\ 0 & 1 - s_3 \end{pmatrix},$$

its operator norm, i.e., the largest singular value equals

$$\begin{aligned} 1 + s_3 &\quad \text{if } s_3 \geq 0, \\ 1 - s_3 &\quad \text{if } s_3 < 0 \end{aligned}$$

we can see that by adding B to identity matrix we can not decrease spectral norm

$$1 = \|I\| \leq \|I + B\| = \|UU^T\|$$

Thus

$$\|U\| \geq 1$$

Algorithm

The following steps lead to a contraction settled by U_{PMNS} and then to its unitary dilation of a minimal dimension

1) Select a finite number of unitary matrices U_i, $i = 1, 2, ...m$, within experimentally allowed range of parameters θ_{13}, θ_{23} and δ.

2) Construct a contraction U_{11} as a convex combination of selected matrices U_i

$$V = \sum_{i=1}^{m} \alpha_i U_i, \quad \alpha_1, ..., \alpha_m \geq 0, \quad \sum_{i=1}^{m} \alpha_i = 1.$$

3) Find singular value decomposition of V, i.e.

$$V = W_1 \Sigma Q_1^\dagger$$

where W_1, Q_1 are unitary, Σ is diagonal, and determine number η of singular values strictly less than 1.

4) Use CS decomposition

$$U = \begin{pmatrix} V & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} =$$

$$\begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \left(\begin{array}{cc|c} I_r & 0 & 0 \\ 0 & C & -S \\ \hline 0 & S & C \end{array} \right) \begin{pmatrix} Q_1^\dagger & 0 \\ 0 & Q_2^\dagger \end{pmatrix}$$

to find the unitary dilation $U \in \mathbb{M}_{(3+\eta)\times(3+\eta)}$ of contraction U_{11}.