Mellin-Barnes representations of Feynman integrals: a construction and solutions

Janusz Gluza (University of Silesia)

in collaboration with DESY Zeuthen team:

Evgen Dubovyk,
Michał Ochman,
Tord Riemann,
Johann Usovitsch

Matter To The Deepest, Ustroń, 14 September 2015

Outline

Introduction

Paul J. Nahin, "Inside interesting integrals", Springer

$$\int_0^1 \frac{1}{[ax+b(1-x)]^2} = \frac{1}{ab}$$

Physics: e.g. $a = 1/(p^2 - m^2)$.

If $ab < 0$, the integral is negative, though integrand is *never* negative

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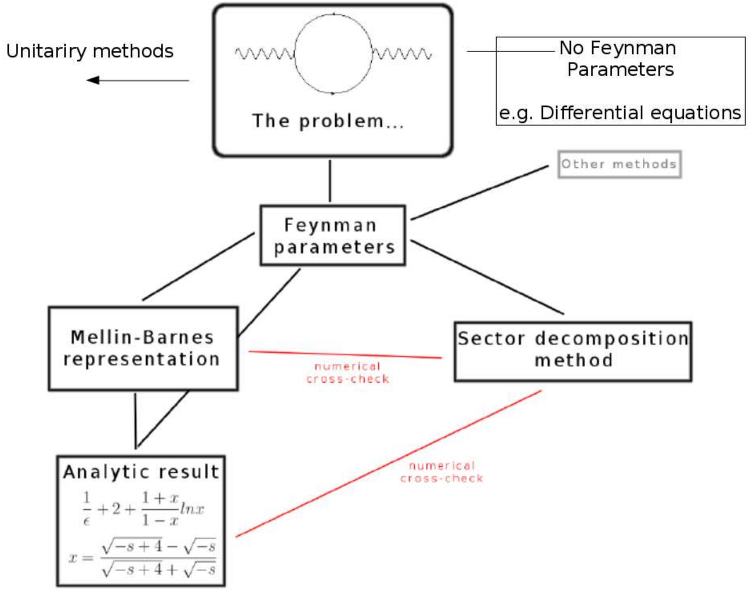
SINGULARITIES

Euclidean vs Minkowskian integrals


Unitariry methods
The problem...
No Feynman Parameters
e.g. Differential equations
Other methods
Feynman parameters
Mellin-Barnes representation
Sector decomposition method
numerical cross-check
numerical cross-check
Analytic result
$\frac{1}{\epsilon} + 2 + \frac{1+x}{1-x} ln x$
$x = \frac{\sqrt{-s+4} - \sqrt{-s}}{\sqrt{-s+4} + \sqrt{-s}}$

L–loop n–point functions

Consider an arbitrary L–loop integral $G(X)$ with loop momenta k_l , with E external legs with momenta p_e and with N internal lines with masses m_i and propagators $1/D_i$

$$G(X) = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \dots d^d k_L \, X(k_1, \dots, k_L)}{D_1^{n_1} \dots D_i^{n_i} \dots D_N^{n_N}}$$

$$d = 4 - 2\epsilon$$

$$D_i = q_i^2 - m_i^2 = \left[\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{M} d_i^e p_e \right] - m_i^2$$

$X(k_1, \dots, k_L)$ stands for tensors in the loop momenta.

Two representations for integrals

Feynman parameter representation ($N_\nu = n_1 + \ldots + n_N$):

$$\frac{1}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} = \frac{\Gamma(n_1 + \ldots + n_N)}{\Gamma(n_1) \ldots \Gamma(n_N)} \int_0^1 dx_1 \ldots \int_0^1 dx_N \frac{x_1^{n_1-1} \ldots x_N^{n_N-1} \delta(1 - x_1 - \ldots - x_m)}{(x_1 D_1 + \ldots + x_N D_N)^{N_\nu}}$$

Alpha parameter representation:

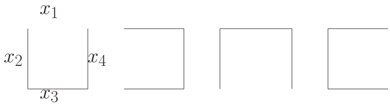
$$\frac{1}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} = \frac{i^{-N_\nu}}{\Gamma(n_1) \ldots \Gamma(n_N)} \int_0^\infty d\alpha_1 \ldots \int_0^\infty d\alpha_N \alpha_1^{n_1-1} \ldots \alpha_N^{n_N-1} e^{i[\alpha_1 D_1 + \ldots + \alpha_N D_N]}$$

For details on equivalence etc, see my talk at "Loops and Legs in QFT" 2014: LL2014.pdf

Starting point for MB

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j \, x_j^{n_j-1}\delta(1-\sum_{i=1}^{N} x_i)\frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

The functions U and F are called graph or Symanzik polynomials.

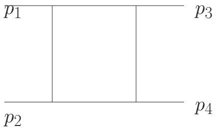


$$U = x_1 + x_2 + x_3 + x_4$$

Trees contributing to the polynomial U for the square diagram

$$F = tx_1x_3 + sx_2x_4$$

$2-$ trees contributing to the polynomial F for the square diagram



Cuts of internal lines such that:

- U: (i) every vertex is still connected to every other vertex by a sequence of uncut lines; (ii) no further cuts without violating (i)
- F: (iii) divide the graph into two disjoint parts such that within each part (i) and (ii) are obeyed and such that at least one external momentum line is connected to each part;

Mellin-Barnes representation

$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda+z)\Gamma(-z) \frac{B^{z}}{A^{\lambda+z}}$$

Mellin-Barnes representation

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$$\frac{1}{(p^2-m^2)^a} = \frac{1}{\Gamma(a)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(a+z)\Gamma(-z) \frac{(m^2)^z}{(p^2)^{a+z}}$$

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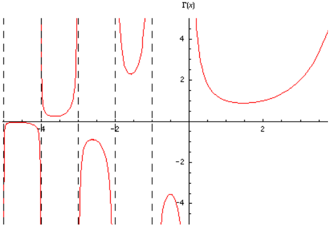
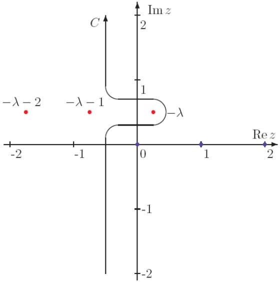
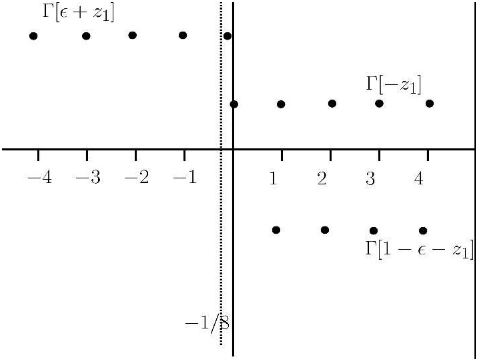
$$\frac{1}{(p^2-m^2)^a} = \frac{1}{\Gamma(a)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(a+z)\Gamma(-z) \frac{(m^2)^z}{(p^2)^{a+z}}$$

Singularities in the complex plane: $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$





Example with more gammas, but 1-dim



Left-half: gives series sutaible for small s

$$\int_{-i\infty}^{+i\infty} dz_1\,(-s)^{-\epsilon}\left(-\frac{m^2}{s}\right)^{z_1}\frac{\Gamma[1-\epsilon-z_1]^2\Gamma[-z_1]\Gamma[\epsilon+z_1]}{\Gamma[2-2\epsilon-z_1]},$$

$$Re[z_1]=-1/8$$

Right-half: gives series suitable for large s

MB integrals and the iterative loop–by–loop (LA) approach

Examples, description, links to basic tools and literature:
http://us.edu.pl/∼ gluza/ambre/

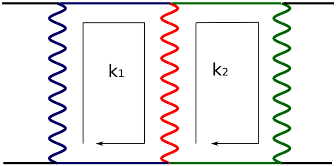


Figure: Loop–by–loop (LA) example

Here: $U(x) \equiv 1$
Input:

$$\mathrm{PR}[\mathrm{k1},\mathrm{m},\mathrm{n1}]\mathrm{PR}[\mathrm{k1}+\mathrm{p1},0,\mathrm{n2}]\mathrm{PR}[\mathrm{k1}+\mathrm{p1}+\mathrm{p2},\mathrm{m},\mathrm{n3}]\mathrm{PR}[\mathrm{k1}-\mathrm{k2},0,\mathrm{n4}]$$
$$\mathrm{PR}[\mathrm{k2},\mathrm{m},\mathrm{n5}]\mathrm{PR}[\mathrm{k2}+\mathrm{p1}+\mathrm{p2},\mathrm{m},\mathrm{n6}]\mathrm{PR}[\mathrm{k2}-\mathrm{p3},0,\mathrm{n7}]$$

Integration over k_2:

$$\mathrm{PR}[\mathrm{k1}-\mathrm{k2},0,\mathrm{n4}]\mathrm{PR}[\mathrm{k2},\mathrm{m},\mathrm{n5}]\mathrm{PR}[\mathrm{k2}+\mathrm{p1}+\mathrm{p2},\mathrm{m},\mathrm{n6}]\mathrm{PR}[\mathrm{k2}-\mathrm{p3},0,\mathrm{n7}]$$

$$\mathrm{F}[\mathrm{X}] = \mathrm{m}^2\,(\mathrm{X}[2]+\mathrm{X}[3])^2 - \mathrm{PR}[\mathrm{k1},\mathrm{m}]\mathrm{X}[1]\mathrm{X}[2] - \mathrm{PR}[\mathrm{k1}+\mathrm{p1}+\mathrm{p2},\mathrm{m}]\mathrm{X}[1]\mathrm{X}[3] \\ -\mathrm{s}\mathrm{X}[2]\mathrm{X}[3] - \mathrm{PR}[\mathrm{k1}-\mathrm{p3},0]\mathrm{X}[1]\mathrm{X}[4]$$

Integration over k_1:

$$\mathrm{PR}[\mathrm{k1},\mathrm{m},\alpha]\mathrm{PR}[\mathrm{k1}+\mathrm{p1},0,\mathrm{n2}]\mathrm{PR}[\mathrm{k1}+\mathrm{p1}+\mathrm{p2},\mathrm{m},\beta]\mathrm{PR}[\mathrm{k1}-\mathrm{p3},0,\gamma]$$

$$\mathrm{F}[\mathrm{X}] = \mathrm{m}^2\,(\mathrm{X}[1]+\mathrm{X}[3])^2 - \mathrm{s}\mathrm{X}[1]\mathrm{X}[3] - \mathrm{t}\mathrm{X}[2]\mathrm{X}[4]$$

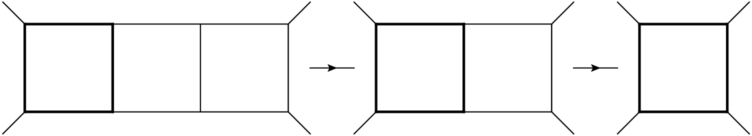
Dimensions of ladder planar MB integrals	Massless cases				Massive cases			
Number of loops (L)	1	2	3	4	1	2	3	4
No Barnes First Lemma	1	4	7	10	3	8	13	18
With BFL	1	4	7	10	2 (1+1)	6 (4+2)	10 (7+3)	14 (10+4)

Optimal results:

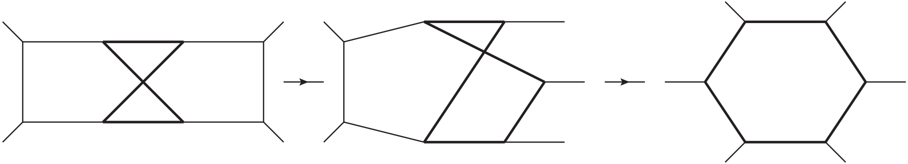
$$\mathrm{Dim}(\mathrm{massive}) = \mathrm{Dim}(\mathrm{massless}) + \#\mathrm{loops}$$

Limitations of LA approach

Planar case:

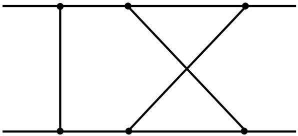


Non-planar case:



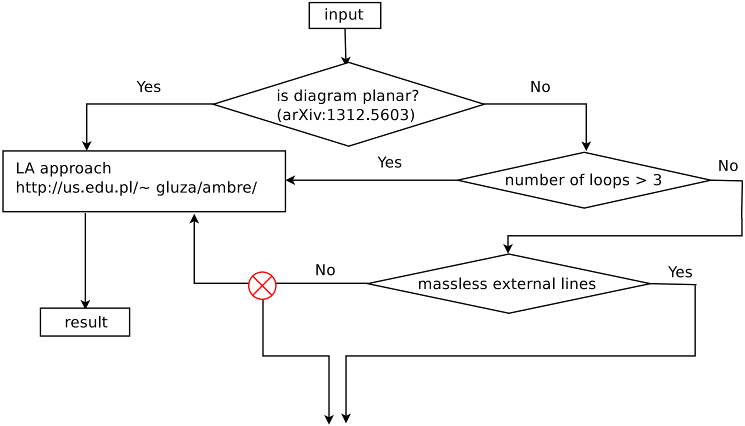
Global approach - GA

Sometimes it is better to change into the MB representation the complete U and F polynomials,
e.g.

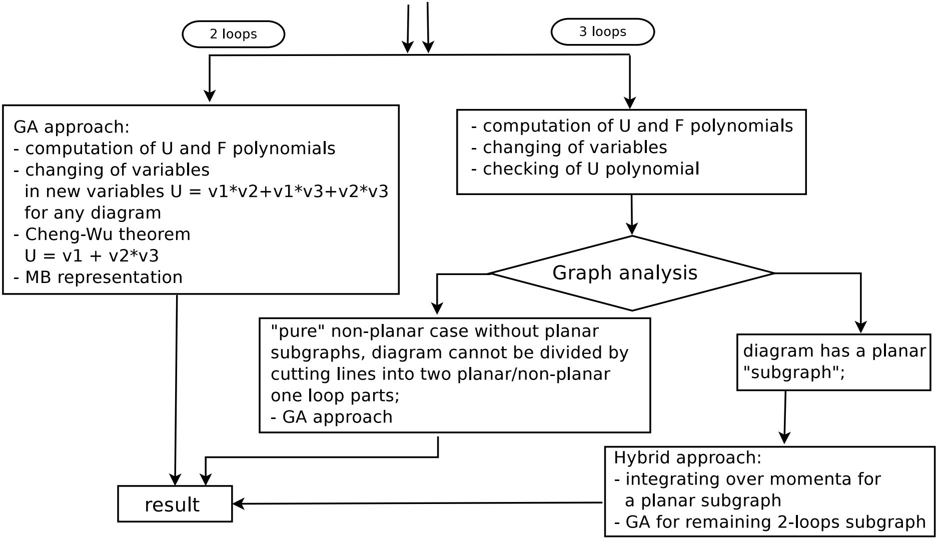


1. massless case: 4-dim MB - GA
2. massive case: 8-dim MB - LA (with GA not less than 10-dim MB (Heinrich, Smirnov, PLB 2004)

1. AMBRE reloaded - non-planar version, basic chart (I)



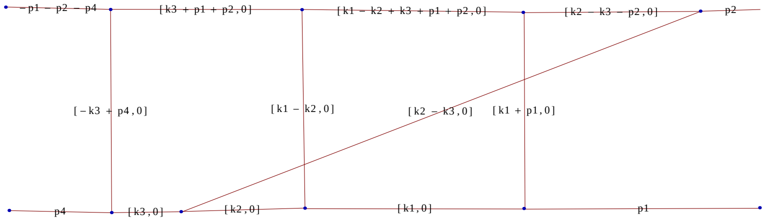
Basic chart (II), GA



```
<< PlanarityTest_v1.1.m
by E. Dubovyk and K. Bielas ver: 1.1
created: January 2014
last executed: 25.04.2014 at 16:48

PlanarityTest[{PR[k1, 0, n1] PR[k1 + p1, 0, n2] ...
               PR[k3, 0, n10] PR[...]}, {k1, k2, k3},
               DrawGraph -> True];
```

Message displayed: The Diagram is non-planar



Cheng–Wu Theorem

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j\, x_j^{n_j-1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

The Cheng–Wu theorem states that the same formula holds with the delta function

$$\delta\left(\sum_{i\in\Omega} x_i - 1\right)$$

where Ω is an arbitrary subset of the lines $1, \ldots, L$, when the integration over the rest of the variables, i.e. for $i \notin \Omega$, is extended to the integration from zero to infinity.
One can prove this theorem in a simple way starting from the alpha representation using

$$1 = \int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1 - \frac{1}{\lambda}\sum_{i=1}^{N}\alpha_i\right) \Leftrightarrow 1 = \int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1 - \frac{1}{\lambda}\sum_{i\in\Omega}\alpha_i\right)$$

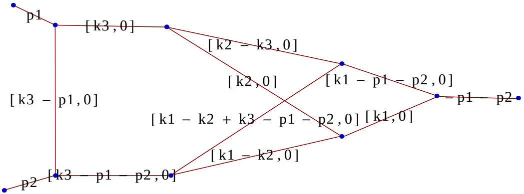
and change variables from α_i to $\alpha_i = \lambda x_i$ as shown above.

Tests

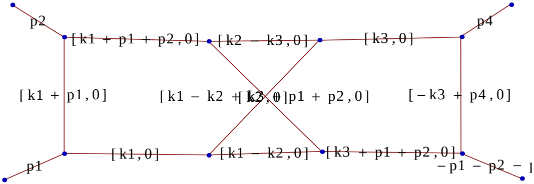
Details go beyond this talk, see basic explanation to my Loops and Legs 2014 talk.

3-loop GA

$U(x) \neq 1$ $U(x)$ is a polinom of degree 3 $Length(U) \gg 1$



U polynomial has 48 terms

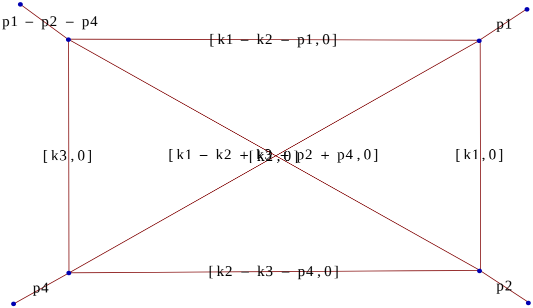


U polynomial has 64 terms

U polynomial for non-planar 3-loop box

```
x[1] x[2] x[4] + x[1] x[3] x[4] + x[2] x[3] x[4] + x[1] x[2] x[5] +
x[1] x[3] x[5] + x[2] x[3] x[5] + x[1] x[4] x[5] + x[2] x[4] x[5] +
x[2] x[4] x[6] + x[3] x[4] x[6] + x[2] x[5] x[6] + x[3] x[5] x[6] +
x[4] x[5] x[6] + x[2] x[4] x[7] + x[3] x[4] x[7] + x[2] x[5] x[7] +
x[3] x[5] x[7] + x[4] x[5] x[7] + x[1] x[2] x[8] + x[1] x[3] x[8] +
x[2] x[3] x[8] + x[1] x[4] x[8] + x[2] x[4] x[8] + x[2] x[6] x[8] +
x[3] x[6] x[8] + x[4] x[6] x[8] + x[2] x[7] x[8] + x[3] x[7] x[8] +
x[4] x[7] x[8] + x[1] x[2] x[9] + x[1] x[3] x[9] + x[2] x[3] x[9] +
x[2] x[4] x[9] + x[3] x[4] x[9] + x[1] x[5] x[9] + x[3] x[5] x[9] +
x[4] x[5] x[9] + x[2] x[6] x[9] + x[3] x[6] x[9] + x[5] x[6] x[9] +
x[2] x[7] x[9] + x[3] x[7] x[9] + x[5] x[7] x[9] + x[1] x[8] x[9] +
x[3] x[8] x[9] + x[4] x[8] x[9] + x[6] x[8] x[9] + x[7] x[8] x[9] +
x[1] x[2] x[10] + x[1] x[3] x[10] + x[2] x[3] x[10] +
x[1] x[4] x[10] + x[2] x[4] x[10] + x[2] x[6] x[10] +
x[3] x[6] x[10] + x[4] x[6] x[10] + x[2] x[7] x[10] +
x[3] x[7] x[10] + x[4] x[7] x[10] + x[1] x[9] x[10] +
x[3] x[9] x[10] + x[4] x[9] x[10] + x[6] x[9] x[10] + x[7] x[9] x[10]
```

Using CW theorem, changes of variables, Barnes Lemmas and some manipulations, we conjecture that for any 3-loop non-planar diagram, U gives only 4 additional integrations
Specific example by Smirnov, arXiv:1312.2588):



agrees with this conjecture.

Final MBrepresentation (6-dim):

```
((-s)^z1 (-t)^(-3 eps-z1-z2) (-u)^z2 Gamma[-z1] Gamma[-z2]
Gamma[3 eps+z1+z2] Gamma[-z3] Gamma[-eps-z2-z4-z5]
Gamma[-z5] Gamma[1-3 eps-z1+z5] Gamma[1- eps+z2+z4+z5]
Gamma[1-3 eps-z1+z3+z4+z5] Gamma[2 eps+z1+z2+z4-z6]
Gamma[-1+4 eps+z1+z2-z3-z4-z5-z6] Gamma[-z6]
Gamma[1-2 eps+z6] Gamma[1-3 eps-z2+z3+z6]
Gamma[1-3 eps-z2-z4+z6] Gamma[1-4 eps-z1-z2+z3+z5+z6])
/(Gamma[2-4 eps] Gamma[1-eps+z2+z4+z5-z6]
Gamma[1-3 eps-z2-z4-z5+z6] Gamma[2-6 eps-z1-z2+z3+z5+z6])
```

Numerical crosscheck $s = t = u = -1$:

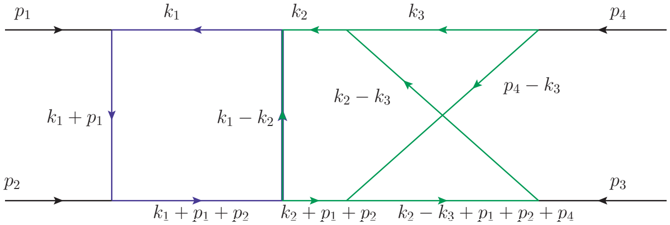
```
AMBRE+MB:  {26.5404 + 2.40412/eps, {0.00580197 + 2.8415*10^-6/eps
FIESTA:    {26.5387 + 2.40417/eps, {0.00021977 + 0.0000206936/eps
```

Some remarks

1. In general, it is not true that $dim(\mathtt{MB}[planars]) \simeq dim(\mathtt{MB}[non-planars])$,
2. Cases of massive external legs are completely different, multiplicity of terms in F is unavoidable

$$F = F_0(\mathrm{scales!}) + \mathrm{U}\sum_{\mathrm{n}=1}^{\mathrm{N}} \mathrm{x_n}\frac{\mathrm{m_n^2}}{\{\mathrm{scales}\}^2}$$

Hybrid method for 3–loops



Hybrid method: LA– $\{k_1\}$; GA– $\{k_2, k_3\}$

- Tord, 2012 in Linz:

  "It would be wonderful to have an algorithm for **automatic evaluation** of all the scalar integrals by infinite sums."

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  "It would be wonderful to have an algorithm for **automatic evaluation** of all the scalar integrals by infinite sums."

- unfortunately, Forest Gump said:

  "Life is like a box a chocolate, you never know what your going to get"

  ... which can be bitter, spicy, etc

— The Mathematica package MBsums transforms MB integrals into sums by Cauchy theorem
— The current version of MBsums is 1.0
— An example. Let

```
int = MBint[-((-x)^(z1 + z6)*y^z2*Gamma[-z1]*
      Gamma[1 + z1]*Gamma[-z1 - z2]*Gamma[-z2]*
      Gamma[z2]*Gamma[-z1 + z2]*Gamma[-z2 - z6]*
      Gamma[z2 - z6]*Gamma[-z6]*Gamma[1 + z6])/
      (2*eps*Gamma[-2*z1]*Gamma[1 - z2]*
      Gamma[1 + z2]*Gamma[-2*z6]), {{eps -> 0},
      {z1 -> -1/2, z2 -> -11/192, z6 -> -1/2}}]
```

Then

```
MBIntToSum[int, {x -> -5, y -> 7},
                {z1 -> L, z6 -> L, z2 -> L}]
```

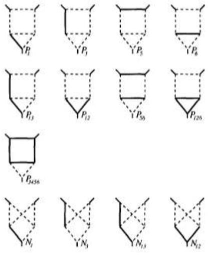
gives

```
{MBsum[-((-x)^(-n1 - n2)*(-1 + n3)!^2*(n1 + n3)!*
      (n2 + n3)!*(2*HarmonicNumber[-1 + n3] -
      2*HarmonicNumber[n3] - HarmonicNumber[
       -1 - n1 + n3] + HarmonicNumber[n1 + n3] -
      HarmonicNumber[-1 - n2 + n3] +
      HarmonicNumber[n2 + n3] - Log[y]))/
      (2*(-1)^(2*n3)*eps*x^2*y^n3*(1 + 2*n1)!*
      (1 + 2*n2)!*n3!^2*(-1 - n1 + n3)!*
      (-1 - n2 + n3)!), n1 >= 0 && n2 >= 0 &&
      n3 >= 1 && 1 + n1 <= n3 && 1 + n2 <= n3,
      {n1, n2, n3}],MBsum[...],...}
```

Nailing 1-dim example

One leg off-shell, $z = q^2/m^2$, $S_a \equiv S_a(n-1) = \sum_{j=1}^{n} 1/j^a$

J. Fleischer et al. / Nuclear Physics B 547 (1999) 343–374 349



vertex diagrams evaluated in this work. Kinematics as in Fig. 1 ($p_1^2 = p_2^2 = 0$). Solid lines
with mass M; dashed lines denote massless propagators.

$$
\begin{aligned}
N_3 = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n (-)^n \Big\{ & \frac{1}{\varepsilon^2}\left[-\tfrac{1}{2}\zeta_2 + K_2\right] \\
& + \frac{1}{\varepsilon}\left[-\tfrac{1}{2}\zeta_3 - 2\zeta_2 S_1 + 2S_3 - 2K_3 + 4S_1 K_2 + (\zeta_2 - S_2)\log(-z)\right] \\
& -\zeta_4 - 2\zeta_3 S_1 - 7\zeta_2 S_2 - 4\zeta_2 S_1^2 + 7\zeta_2 K_2 - \tfrac{7}{2} S_4 + \tfrac{7}{2} S_2^2 + 6 S_1 S_3 + 2 S_{1,3} + 8 K_4 \\
& -8 S_1 K_3 + 8 S_1^2 K_2 + \left(\zeta_3 + 4\zeta_2 S_1 - S_{1,2} - 3 S_1 S_2 - 4 K_3\right)\log(-z) \\
& + \left(-\zeta_2 + \tfrac{1}{2} S_2 + K_2\right)\log^2(-z) \Big\}, \qquad \text{(D.11)}
\end{aligned}
$$

Hardly converging for $z = 1$, even for 3000 terms, and poor precision

Numerical way - MBnumerics package for Minkowskian region

One-loop:

- FeynArts, LoopTools (FF package)
- OneLoop, QCDLoop, ...
- MadGraph, Sherpa, Powheg-Box, Helac-NLO, Golem, GoSam, ...
- SecDec, Fiesta, CSectors, sector_decomposition, ...

More general methods:

- tree-duality
- unitarity
- contour deformations
- deqs, expansions by regions,...

MB:

- M. Czakon, Automatized analytic continuation of Mellin-Barnes integrals, Comput.Phys.Commun. 175 (2006) 559
- Ayres Freitas, Yi-Cheng Huang, On the Numerical Evaluation of Loop Integrals With Mellin-Barnes Representations, JHEP 1004 (2010) 074
- phase space integrations with MB: G. Somogyi, Z. Trocsanyi...
- transforming MB integrals into Dirac delta constraints, Anastasiou et al, arXiv:1302.4379 (appendix C, parametric integrals instead of nested sums)

General structure of the MB integrals after expansion in ϵ

$$\frac{1}{(2\pi i)^r}\int_{c_1-i\infty}^{c_1+i\infty}\dots\int_{c_r-i\infty}^{c_r i\infty}\prod_i^r dz_i\ \mathbf{F}(Z,S)\,\frac{\prod_j \mathbf{G_j}(N_j)}{\prod_k \mathbf{G_k}(N_k)}$$

F depends on: Z – linear combinations of r complex variables z_i,
S – kinematic parameters and masses;

$\mathbf{G_i}$: Gamma and PolyGamma functions
N_i : linear combinations of z_i, e.g. $N_i = \sum_l \alpha_{il} z_l + \gamma_i$

In practice F is a product of powers of S:

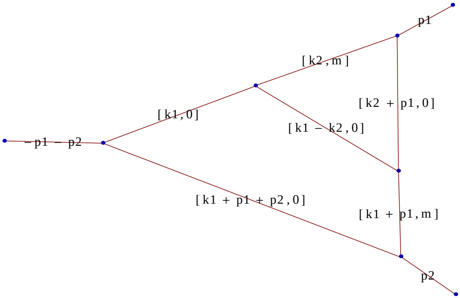
$$\mathbf{F} \sim \prod_k X_k^{\sum_i(\alpha_{ki}z_i+\gamma_k)}$$

$$\alpha_{ij},\gamma_i \in \text{Integer},\quad X=\left\{-\frac{s}{m_1^2},\frac{m_1^2}{m_2^2},\frac{s}{t},\dots\right\}.$$

In Minkowskian region $S>0$. Let's consider a case with $X=-\frac{s}{m^2}$ and $s=m^2$.

Where is the problem?

An Example:




```
{MBint[((-1)^(-z1) Gamma[-1-z1] Gamma[2+z1] Gamma[-1-z2]
 Gamma[z1-z2] Gamma[1+z2-z3]^2 Gamma[-z3] Gamma[1+z3]
Gamma[-z1+z3]^2 Gamma[-z2+z3])/(Gamma[-z1] Gamma[1+z1-z2]
          Gamma[1-z1+z3]), {{eps -> 0},
    {z1 -> -47/37, z2 -> -139/94, z3 -> -176/235}}]}
```

MB.m, integration in Euclidean region

Steps for numerical integration:

- real parametrization $z_i \to c_i + It_i, \quad t_i \in (-\infty, \infty)$
- **MB.m way:** transformation to finite integration interval (here $[0,1]$ and linked with CUBA library)

$$t_i \to Log\left[\frac{x_i}{1-x_i}\right] \quad dt_i \to \frac{dx_i}{x_i(1-x_i)}$$

In general, the factor $(-1)^{-z1} \to e^{-\pi t_1}$
may lead to problems in asymptotic limit $t_1 \to -\infty$.
Fortunately, this factor cancels with remaining gammas, taking

$$\lim_{t \to \pm\infty} \Gamma(a + It) \sim e^{-\frac{\pi|t|}{2}} t^{a-\frac{1}{2}}$$

and a ray $t_1 = t, \; t_2 = 0, \; t_3 = 0$ we can compute a limit for a product of gamma functions:

$$\frac{\prod_j \mathbf{G_j}(N_j)}{\prod_k \mathbf{G_k}(N_k)} \sim e^{\pi t} \frac{1}{t^{646/235}}$$

What remains is $\sim \frac{1}{t^\alpha}$.

However, transformation to the finite region does not remove singular behavior of the integrand

$$\frac{1}{x\,Log[x]^{646/235}} \xrightarrow{x\to 0} \infty$$

This kind of sigularity is integratable when $\alpha > 1$ (our example)
The solution in general is to take **another transformation**

$$t_i \to Tan[\pi(x_i - \frac{1}{2})] \quad dt_i \to \frac{\pi dx_i}{Cos[\pi(x_i - \frac{1}{2})]^2}$$

Now we have

$$\frac{Cos[\pi(x_i - \frac{1}{2})]^{176/235}}{Sin[\pi(x_i - \frac{1}{2})]^{646/235}} \xrightarrow{x\to 0} 0$$

With yet another trick concerning a product of gamma functions ($LogG$ denotes log-gamma function, see CernLib documentation).

$$\frac{\prod_j \mathbf{G_j}(N_j)}{\prod_k \mathbf{G_k}(N_k)} = Exp\left(\sum_j LogG_j(N_j) - \sum_k LogG_k(N_k)\right)$$

we have a chance to get stable results:

```
Analytical:         -1.199526183135566 + 5.567365907880696 I
Our MBnumerics:     -1.199526183168498 + 5.567365907904922 I
MB(Vegas):          -1.199561086311856 + 5.569395048002913 I
MB(Cuhre):                               NaN
FIESTA:             -1.200370278497323 + 5.561435923863947 I
SecDec:                                no output
```

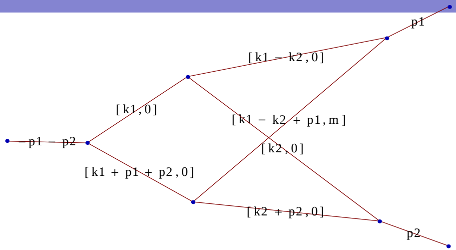
"Euclidean" results:

```
Analytical result: 3.376807975550548
MB(Vegas):         3.376922163980158
MB(Cuhre):         3.376807975447292
FIESTA:            3.376815834907247
SecDec:                 big error
```

Remark: For integrals where α which defines singular behaviour of the integrand is smaller than 1,

$$\frac{\prod_j \mathbf{G_j}(N_j)}{\prod_k \mathbf{G_k}(N_k)} \sim \frac{1}{t^\alpha}$$

yet another smart trick must be done ...



Results (constant part of the MBintegrals):

```
Analytical:       -0.778599608979684 - 4.123512593396311 I
Our MBnumerics:   -0.778599608324769 - 4.123512600516016 I
MB(Vegas):                             big error
MB(Cuhre):                                NaN
FIESTA:                                big error
SecDec:                                no output
```

Euclidean results (constant part):

```
Analytical:         -0.4966198306057021
MB(Vegas):          -0.4969417442183914
MB(Cuhre):          -0.4966198313219404
FIESTA:             -0.4966184488196595
SecDec:             -0.4966192150541896
```

Summary

- a construction of MB integrals is optimized well up to two-loops, more complicated at three-loops
- usually high dimensional integrals for variety of masses, legs, loops involved,
- solving analytically MB integrals through nested sums - hard thing
- numerical approach very promising, first package already available internally which allows for physical applications beyond present capabilities - more at LL 2016.