Mellin-Barnes representations of Feynman integrals: a construction and solutions

Janusz Gluza (University of Silesia)

in collaboration with DESY Zeuthen team:

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Johann Usovitsch

Kanpur, 22 February 2016

Outline

1 Introduction

2 Mellin-Barnes Feynman integrals

3 Analytical and numerical results

4 Extracting singularities - some details

5 Summary

6 Backup material

Use of basic science

WHAT'S THE USE OF BASIC SCIENCE?

Christopher Llewellyn Smith,
Director-General of CERN from 1994-1998

by C.H. Llewellyn Smith,
former Director-General of CERN
Original: The use of basic science

Content:

1. Introduction
2. Basic versus applied science
3. Benefits of basic science
4. Why governments must support basic science
5. Can it be left to others? Lessons from Japan?
6. What science to fund
7. Concluding remarks

Contributions to the Culture

Bob Wilson (first Director of Fermilab), when asked by a Congressional Committee:

Q: "What will your lab contribute to the defence of the US?"

A: "Nothing, but it will make it worth defending".

Talk by Johannes Bluemlein, Radcor 2011, India

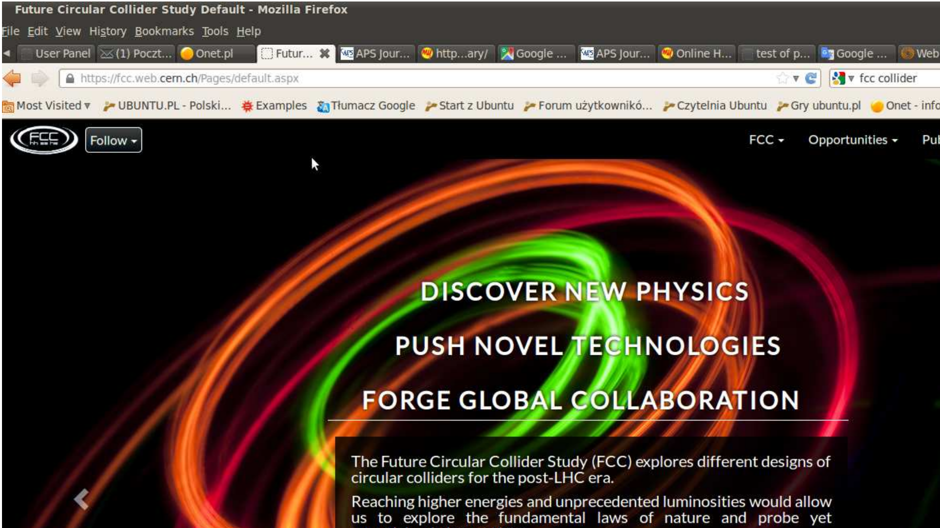
Why Precision ? Multi-Leg Processes Resummations and Infrared Structure Precision Calculations for Low-Energy Processes Multi-Loop Corrections Math

Why Precision ? (in experiment and theory)

- T. Brahe (-1601) Detailed measurement of planetary motion. Rudolphine Tables;
  $\Longrightarrow$ J. Kepler (1609, 1618) Laws of planetary orbit;
  $\Longrightarrow$ I. Newton (1686) Classical Gravity.
- A. Michelson (1881) The velocity of light is constant (in flat space).
  $\Longrightarrow$ A. Einstein (1905) Special Relativity. [Possible exception: arXiv:1109.4897].
- O. Lummer, E. Pringsheim, H. Rubens, and F. Kurlbaum (1900) measure precisely the Spectrum of the black body radiation.
  $\Longrightarrow$ M. Planck (1900) Quantization of the action.
- O. Stern and O. Frisch (1933) Anomalous magnetic moment of the proton.
  $\Longrightarrow$ SLAC-MIT experiments (1968/69), final discovery:
  The proton consists of quark-gluon partons.
- C. Prescott et. al. (1979) Electro-weak asymmetry in DIS γ-Z interference.
  UA1, UA2 (1983) Production of W and Z bosons.
- M. Veltman (1977) ρ-parameter depends on m_t^2/m_b^2; $\Longleftarrow$SCHOONSHIP
  $\Longrightarrow$ Tevatron (1994/95) Top discovery.
- Precise Loop corrections to the Standard Model
- Constraints on the Higgs Boson

DESY

Needs for precise calculations: LHC, FCC/ILC/ILC/CEPC, BEPC/PEP/VEPP,...



Needs for precise calculations: LHC, FCC/ILC/ILC/CEPC, BEPC/PEP/VEPP,...

https://fcc.web.cern.ch/Pages/Collaboration.aspx

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Follow FCC Opportunities Publi

There is growing evidence that large-scale scientific infrastructures are best built and operated in a sustainable manner as a worldwide joint effort. The FCC study forms an open international collaboration, aiming at a geographically well-balanced and topically complementary network of contributions. This structure federating resources worldwide forms the core of a globally coordinated strategy of converging activities, involving participants from the ERA and beyond. Organisations from North America, South America and Asia are invited to participate in order to lay the foundations for subsequent development actions that will strengthen the ERA as a focal point of global research cooperation.

This inclusive approach embracing the worldwide science and technology community both in an open and incremental participation process is the first step to form an international platform for the realization of a next generation, frontier particle physics research infrastructure, leveraging existing assets and available experience.

Participating Institutes

Austria

- *TUWIEN*, Vienna
- *HEPHY*, Wien

Belarus

- *NC PHEP BSU*, Minsk
- *INP BSU*, Minsk

Brasil

- *CBPF*, Rio de Janeiro

India

- *IITK*, Kanpur

Iran

- *IPM*, Tehran

Italy

- *UNIMI*, Milan
- *Sapienza*, Rome
- *INFN*, Frascati (Roma)

Serbia

- *UB*, Belgrade

South Korea

- *KUS*, Sejong
- *KU*, Seoul
- *KIAS*, Seoul
- *KAIST*, Yuseong-gu
- *GWNU*, Wonju-Si

Introduction

Paul J. Nahin, "Inside interesting integrals", Springer

$$\int_0^1 \frac{1}{[ax+b(1-x)]^2} = \frac{1}{ab}$$

Physics: e.g. $a = 1/(p^2 - m^2)$.

If $ab < 0$, the integral is negative, though integrand is *never* negative

Introduction

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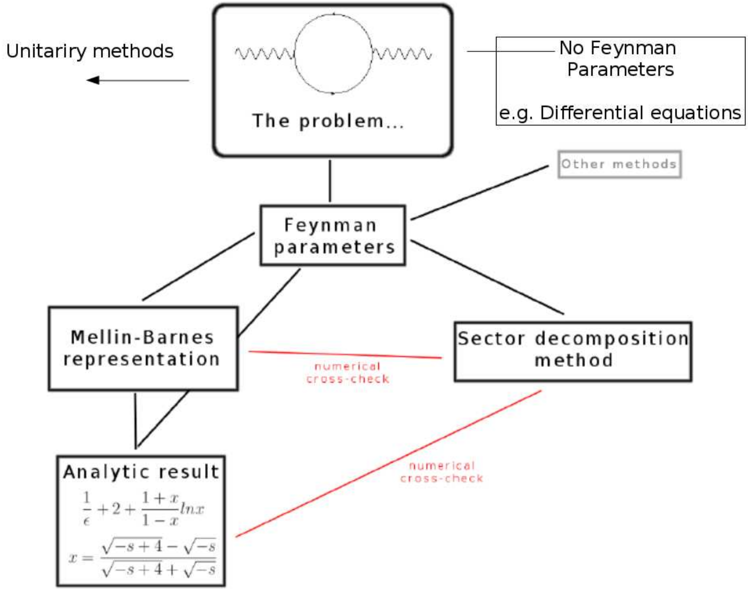
SINGULARITIES

Euclidean vs Minkowskian integrals


Unitariry methods
The problem...
No Feynman
Parameters
e.g. Differential equations
Other methods
Feynman
parameters
Mellin-Barnes
representation
Sector decomposition
method
numerical
cross-check
Analytic result
$\frac{1}{\epsilon}+2+\frac{1+x}{1-x}ln x$
$x=\frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}}$
numerical
cross-check

Methods

One-loop:

- FeynArts, LoopTools (FF package)
- OneLoop, QCDLoop, ...
- MadGraph, Sherpa, Powheg-Box, Helac-NLO, Golem, GoSam, ...
- SecDec, Fiesta, CSectors, sector_decomposition, ...

More general methods:

- tree-duality
- unitarity
- contour deformations
- deqs, expansions by regions,...

Mellin-Barnes (MB):

- M. Czakon, Automatized analytic continuation of Mellin-Barnes integrals, Comput.Phys.Commun. 175 (2006) 559
- Ayres Freitas, Yi-Cheng Huang, On the Numerical Evaluation of Loop Integrals With Mellin-Barnes Representations, JHEP 1004 (2010) 074
- phase space integrations with MB: G. Somogyi, Z. Trocsanyi...
- transforming MB integrals into Dirac delta constraints, Anastasiou et al, arXiv:1302.4379 (appendix C, parametric integrals instead of nested sums)

CSectors web page

http://prac.us.edu.pl/ gluza/csectors/

prac.us.edu.pl/~gluza/csectors/

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CSectors - numerical calculation of multiloop tensor integrals in Euclidean region by sector decomposition

arXiv: arXiv:1006.4728 [hep-ph] (basic description)
arXiv: arXiv:1010.1667 [hep-ph] (the main paper)

J. Gluza, K. Kajda (Silesia U.) , T. Riemann, V. Yundin (DESY, Zeuthen)

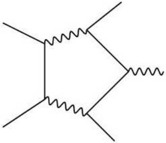
See here (Mellin-Barnes) for an alternative way of numerical calculation of Feynman Integrals in Euclidean region.

To download 'right click' and 'save target as'.

- **The package CSectors.m, version 1.0**
  The package compute numerically the Laurent expansion of (divergent) multi-loop tensor Feynman integrals. It generates appropriate files in an automatic way and link them with the basic sector decomposition program by Bogner and Weinzierl webpage. See there for description and download of the package altogether with Ginac
  Detailed description how to define integrals for numerical calculations can be found in the following Mathematica notebook examples:
  - numerical_checks.nb

Tarball with another sample examples given below ex.tgz. Some of examples below correspond to the examples calculated by Mellin-Barnes on the webpage here .

- SD_pentagon.sh, output_pentagon
  - Massive QED pentagon diagram.



- SD_1lbox.sh, output_1lbox
  - Massive QED one-loop box diagram.

AMBRE web page

http://prac.us.edu.pl/ gluza/ambre/

AMBRE - Automatic Mellin-Barnes REpresentation

First paper, ver. 1.0: Computer Physics Communications 177 (2007) 879.
Second paper, ver. 2.0: arXiv: arXiv:1010.1667 [hep-ph].

J. Gluza, K. Kajda (Silesia U.) , T. Riemann (DESY, Zeuthen)

See here (sectors) for an alternative way of numerical calculation of Feynman Integrals in Euclidean region.

To download 'right click' and 'save target as'.

- **The package AMBRE.m, version 2.0**
  arXiv: arXiv:1006.4728 [hep-ph] (basic description), arXiv: arXiv:1010.1667 [hep-ph] (main description).
  This version allows to generate in an automatic way M-B representations for multiloop planar tensor integrals. If you want to control manually loop-by-loop dimensionality of constructed M-B representations, use one of previous versions.
  To make analytic continuation and numerical tests, it needs auxiliary file MBnum.m or MBresolve.m (by Alex and Volodya Smirnov) and to reduce dimensionality of integrals as much as possible the package barnesroutines.m (by David Kosower) can be useful. Examples:
  - MBnum used: MB_SE5l0m.m, out_SE5l0m
    MBresolve used: MB_SE5l0m_MBresolve.m, out_SE5l0m_MBresolve
    MB_B1_massless.m, out_B1_massless
    MB_B1_massive.m, out_B1_massive
    barnesroutines used: MB_B1_massive_BL.m, out_B1_massive_BL
    Rank 2 cases: MB_B1_massive_rank2.m, out_B1_massive_rank2
    MB_B1_massless_rank2.m, out_B1_massless_rank2
  - Pentabox of rank 3 (for a picture see example10.nb below): MB_PBox.m, out_PBox
  - Four loop self-energy (for a picture see example9.nb below): MB_SE4loop.m, out_SE4loop
  - tar file with all above examples - ex.tgz
- **The package AMBREv1.2.m, version 1.2**
  This version allows to generate M-B representations for tensor integrals containing not only scalar products of internal and external momenta, but also internal momenta with indices only. Additionally new options were added, among others it allows to generate representations without doing "X" integration (here we would like to thank Pierpaolo Mastrolia for this suggestion). Detailed description of new features is avaliable in the following examples:

More material

http://prac.us.edu.pl/ gluza/capp2013/

Files and links, for a background
see also lectures by Tord Riemann: pdf

- Plan for tutorials: pdf
- Web page for AMBRE
- MBtools: link
  (download MB.m by M. Czakon and MBresolve.m by Smirnov&Smirnov and install Cuba by T. Hahn)
- Sector decomposition, available public programs:

  (i) Bogner-Weinzierl (BW) sector_decomposition

  (ii) Mathematica CSectors interface to the Ginac BW sector_decomposition

  (iii) Smirnov/Tentioukov, FIESTA

  (iv) Borowka/Carter/Heinrich, SecDec,
- Available public programs for IBPs:

  (i) FIRE

  (ii) Reduze
- HPLs: link
- tgz file with tutorial files (download from Katowice): capp2013.tgz
  fiesta2.tgz

L–loop n–point functions

Consider an arbitrary L–loop integral $G(X)$ with loop momenta k_l , with E external legs with momenta p_e and with N internal lines with masses m_i and propagators $1/D_i$

$$G(X) = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \ldots d^d k_L \, X(k_1, \ldots, k_L)}{D_1^{n_1} \ldots D_i^{n_i} \ldots D_N^{n_N}}$$

$$d = 4 - 2\epsilon$$

$$D_i = q_i^2 - m_i^2 = \left[\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{M} d_i^e p_e \right] - m_i^2$$

$X(k_1, \ldots, k_L)$ stands for tensors in the loop momenta.

Two representations for integrals

Feynman parameter representation ($N_\nu = n_1 + \ldots + n_N$):

$$\frac{1}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} = \frac{\Gamma(n_1 + \ldots + n_N)}{\Gamma(n_1) \ldots \Gamma(n_N)} \int_0^1 dx_1 \ldots \int_0^1 dx_N \frac{x_1^{n_1-1} \ldots x_N^{n_N-1} \delta(1 - x_1 - \ldots - x_m)}{(x_1 D_1 + \ldots + x_N D_N)^{N_\nu}}$$

Alpha parameter representation:

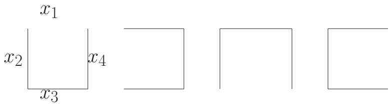
$$\frac{1}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} = \frac{i^{-N_\nu}}{\Gamma(n_1) \ldots \Gamma(n_N)} \int_0^\infty d\alpha_1 \ldots \int_0^\infty d\alpha_N \alpha_1^{n_1-1} \ldots \alpha_N^{n_N-1} e^{i[\alpha_1 D_1 + \ldots + \alpha_N D_N]}$$

For details on equivalence etc, see my talk at "Loops and Legs in QFT" 2014: LL2014.pdf

Starting point for MB

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j\, x_j^{n_j-1}\delta(1-\sum_{i=1}^{N}x_i)\frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

The functions U and F are called graph or Symanzik polynomials.

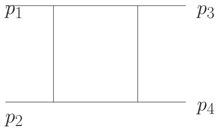


$$U = x_1 + x_2 + x_3 + x_4$$

Trees contributing to the polynomial U for the square diagram

$$F = tx_1x_3 + sx_2x_4$$

2 − trees contributing to the polynomial F for the square diagram



Cuts of internal lines such that:

- U: (i) every vertex is still connected to every other vertex by a sequence of uncut lines; (ii) no further cuts without violating (i)
- F: (iii) divide the graph into two disjoint parts such that within each part (i) and (ii) are obeyed and such that at least one external momentum line is connected to each part;

Some Definitions

- Spanning tree T for the graph G
  sub-graph with the following properties:
  - T contains all the vertices of G
  - the number of loops in T is zero
  - T is connected

  T can be obtained from G by deleting L edges (L – number of loops in G)
- Spanning k–forest F for the graph G
  has the same properties as T but it is not required that a spanning forest is connected
  F can be obtained from G by deleting $L + k - 1$ edges

If $\mathcal{T}$ is the set of spanning forests of G and $\mathcal{T}_k$ is set of s panning k-forests of G when

$$\mathcal{T} = \bigcup_{k=1}^{r} \mathcal{T}_k \quad (r - \mathrm{numberofvertices})$$

($\mathcal{T}_k$ is the set of spanning trees)
Each element of $\mathcal{T}_k$ has k connected components $(T_1, \ldots, T_k)$
P_{T_i} is the set of external momenta attached to T_i for a given k–forest.

The spanning trees and the spanning 2–forests of a graph G are closely related to the graph polynomials U and F of the graph:

$$U = \sum_{T \in \mathcal{T}_1} \prod_{e_i \notin T} x_i$$

$$F = -\sum_{(T_1,T_2)\in \mathcal{T}_2} \left(\prod_{e_i \notin (T_1,T_2)} x_i \right) \left(\sum_{p_i \in P_{T_1}} p_i \right) \left(\sum_{p_j \in P_{T_2}} p_j \right) + U \sum_{i=1}^{n} x_i m_i^2 = F_0 + U \sum_{i=1}^{n} x_i m_i^2$$

Example:

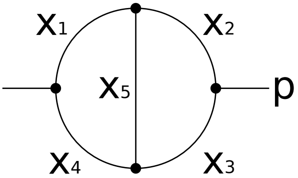


Figure 1: A two-loop two-point graph.

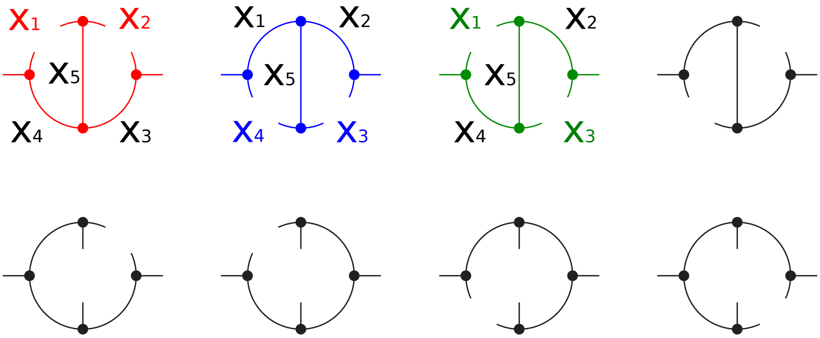


Figure 2: The set of spanning trees for the two-loop two-point graph of fig. 4.

$$U = x_1x_2 + x_3x_4 + x_1x_3 + x_2x_4 + x_1x_5 + x_2x_5 + x_3x_5 + x_4x_5$$

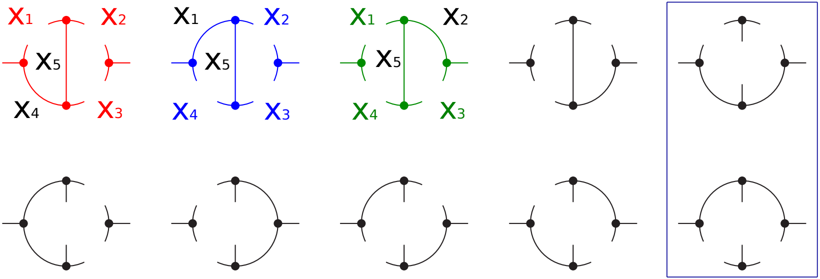


Figure 3: The set of spanning 2-forests for the two-loop two-point graph of fig. 4.

$$F = [x_1x_2x_3 + x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4 + x_2x_3x_5 + x_1x_4x_5 + x_2x_4x_5 + x_1x_3x_5]\,(-p^2)$$

Mellin-Barnes representation ("One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane" - JS)

$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda+z)\Gamma(-z) \frac{B^{z}}{A^{\lambda+z}}$$

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$$\frac{1}{(p^2-m^2)^a} = \frac{1}{\Gamma(a)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(a+z)\Gamma(-z) \frac{(m^2)^z}{(p^2)^{a+z}}$$

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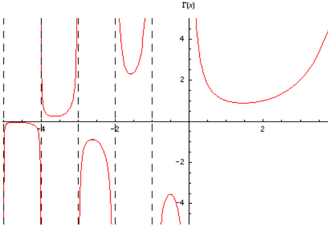
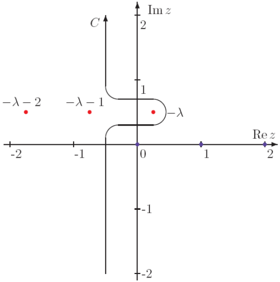
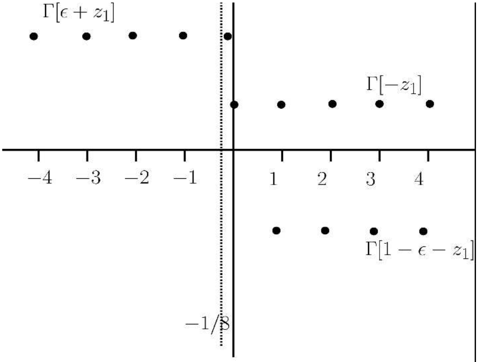
Singularities in the complex plane: $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$





Example with more gammas, but 1-dim



Left-half: gives series sutaible for small s

$$\int_{-i\infty}^{+i\infty} dz_1\, (-s)^{-\epsilon}\left(-\frac{m^2}{s}\right)^{z_1}\frac{\Gamma[1-\epsilon-z_1]^2\Gamma[-z_1]\Gamma[\epsilon+z_1]}{\Gamma[2-2\epsilon-z_1]},$$

$$Re[z_1] = -1/8$$

Right-half: gives series suitable for large s

MB integrals and the iterative loop–by–loop (LA) approach

Examples, description, links to basic tools and literature:
http://us.edu.pl/∼ gluza/ambre/

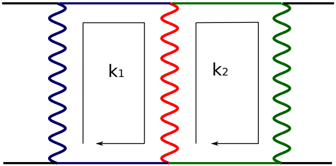


Figure 4: Loop–by–loop (LA) example

Here: $U(x) \equiv 1$
Input:

$$\mathrm{PR}[\mathrm{k1}, \mathrm{m}, \mathrm{n1}]\mathrm{PR}[\mathrm{k1}+\mathrm{p1}, 0, \mathrm{n2}]\mathrm{PR}[\mathrm{k1}+\mathrm{p1}+\mathrm{p2}, \mathrm{m}, \mathrm{n3}]\mathrm{PR}[\mathrm{k1}-\mathrm{k2}, 0, \mathrm{n4}]$$
$$\mathrm{PR}[\mathrm{k2}, \mathrm{m}, \mathrm{n5}]\mathrm{PR}[\mathrm{k2}+\mathrm{p1}+\mathrm{p2}, \mathrm{m}, \mathrm{n6}]\mathrm{PR}[\mathrm{k2}-\mathrm{p3}, 0, \mathrm{n7}]$$

Integration over k_2:

$$\mathrm{PR}[\mathrm{k1}-\mathrm{k2},0,\mathrm{n4}]\mathrm{PR}[\mathrm{k2},\mathrm{m},\mathrm{n5}]\mathrm{PR}[\mathrm{k2}+\mathrm{p1}+\mathrm{p2},\mathrm{m},\mathrm{n6}]\mathrm{PR}[\mathrm{k2}-\mathrm{p3},0,\mathrm{n7}]$$

$$\mathrm{F}[\mathrm{X}] = \mathrm{m}^2\left(\mathrm{X}[2]+\mathrm{X}[3]\right)^2 - \mathrm{PR}[\mathrm{k1},\mathrm{m}]\mathrm{X}[1]\mathrm{X}[2] - \mathrm{PR}[\mathrm{k1}+\mathrm{p1}+\mathrm{p2},\mathrm{m}]\mathrm{X}[1]\mathrm{X}[3] - \mathrm{s}\mathrm{X}[2]\mathrm{X}[3] - \mathrm{PR}[\mathrm{k1}-\mathrm{p3},0]\mathrm{X}[1]\mathrm{X}[4]$$

Integration over k_1:

$$\mathrm{PR}[\mathrm{k1},\mathrm{m},\alpha]\mathrm{PR}[\mathrm{k1}+\mathrm{p1},0,\mathrm{n2}]\mathrm{PR}[\mathrm{k1}+\mathrm{p1}+\mathrm{p2},\mathrm{m},\beta]\mathrm{PR}[\mathrm{k1}-\mathrm{p3},0,\gamma]$$

$$\mathrm{F}[\mathrm{X}] = \mathrm{m}^2\left(\mathrm{X}[1]+\mathrm{X}[3]\right)^2 - \mathrm{s}\mathrm{X}[1]\mathrm{X}[3] - \mathrm{t}\mathrm{X}[2]\mathrm{X}[4]$$

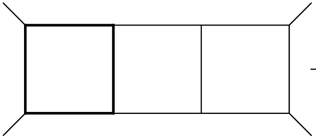
Dimensions of ladder planar MB integrals	Massless cases				Massive cases			
Number of loops (L)	1	2	3	4	1	2	3	4
No Barnes First Lemma	1	4	7	10	3	8	13	18
With BFL	1	4	7	10	2 (1+1)	6 (4+2)	10 (7+3)	14 (10+4)

Optimal results:

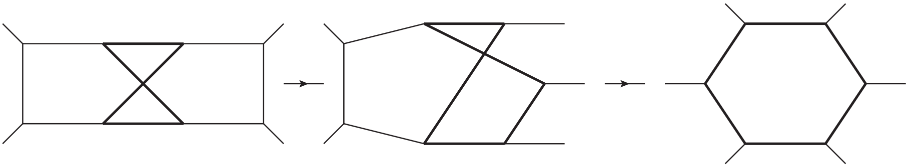
$$\mathrm{Dim(massive)} = \mathrm{Dim(massless)} + \#\mathrm{loops}$$

Limitations of LA approach

Planar case:

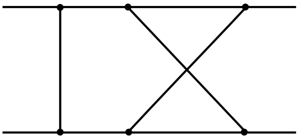


Non-planar case:



Global approach - GA

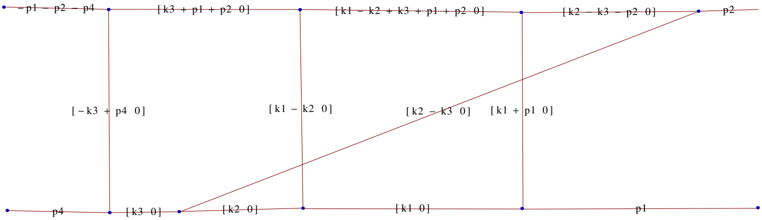
Sometimes it is better to change into the MB representation the complete U and F polynomials,
e.g.



1. massless case: 4-dim MB - GA
2. massive case: 8-dim MB - LA (with GA not less than 10-dim MB (Heinrich, Smirnov, PLB 2004)

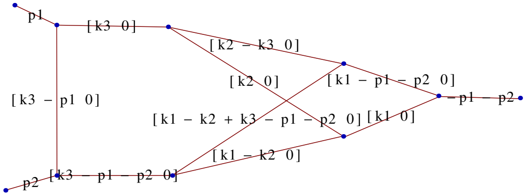
```
<< PlanarityTest_v1.1.m
by E. Dubovyk and K. Bielas ver: 1.1
created: January 2014
last executed: 25.04.2014 at 16:48

PlanarityTest[{PR[k1, 0, n1] PR[k1 + p1, 0, n2] ...
               PR[k3, 0, n10] PR[...]}, {k1, k2, k3},
               DrawGraph -> True];
```

Message displayed: The Diagram is non-planar

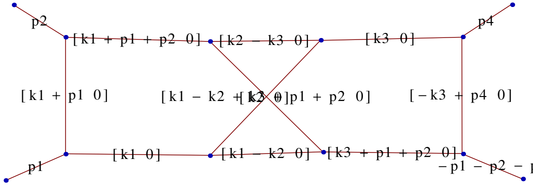


3-loop GA

$U(x) \neq 1 \quad U(x)$ is a polinom of degree 3 $\quad Length(U) \gg 1$



U polynomial has 48 terms

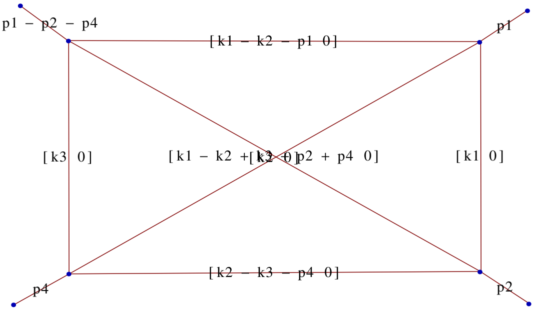


U polynomial has 64 terms

U polynomial for non-planar 3-loop box

```
x[1] x[2] x[4] + x[1] x[3] x[4] + x[2] x[3] x[4] + x[1] x[2] x[5] +
x[1] x[3] x[5] + x[2] x[3] x[5] + x[1] x[4] x[5] + x[2] x[4] x[5] +
x[2] x[4] x[6] + x[3] x[4] x[6] + x[2] x[5] x[6] + x[3] x[5] x[6] +
x[4] x[5] x[6] + x[2] x[4] x[7] + x[3] x[4] x[7] + x[2] x[5] x[7] +
x[3] x[5] x[7] + x[4] x[5] x[7] + x[1] x[2] x[8] + x[1] x[3] x[8] +
x[2] x[3] x[8] + x[1] x[4] x[8] + x[2] x[4] x[8] + x[2] x[6] x[8] +
x[3] x[6] x[8] + x[4] x[6] x[8] + x[2] x[7] x[8] + x[3] x[7] x[8] +
x[4] x[7] x[8] + x[1] x[2] x[9] + x[1] x[3] x[9] + x[2] x[3] x[9] +
x[2] x[4] x[9] + x[3] x[4] x[9] + x[1] x[5] x[9] + x[3] x[5] x[9] +
x[4] x[5] x[9] + x[2] x[6] x[9] + x[3] x[6] x[9] + x[5] x[6] x[9] +
x[2] x[7] x[9] + x[3] x[7] x[9] + x[5] x[7] x[9] + x[1] x[8] x[9] +
x[3] x[8] x[9] + x[4] x[8] x[9] + x[6] x[8] x[9] + x[7] x[8] x[9] +
x[1] x[2] x[10] + x[1] x[3] x[10] + x[2] x[3] x[10] +
x[1] x[4] x[10] + x[2] x[4] x[10] + x[2] x[6] x[10] +
x[3] x[6] x[10] + x[4] x[6] x[10] + x[2] x[7] x[10] +
x[3] x[7] x[10] + x[4] x[7] x[10] + x[1] x[9] x[10] +
x[3] x[9] x[10] + x[4] x[9] x[10] + x[6] x[9] x[10] + x[7] x[9] x[10]
```

Using Chang-Wu theorem, changes of variables, Barnes Lemmas and some manipulation
we conjecture that for any 3-loop non-planar diagram, U gives only 4 additional integrations
Specific example by Smirnov, arXiv:1312.2588):



agrees with this conjecture.

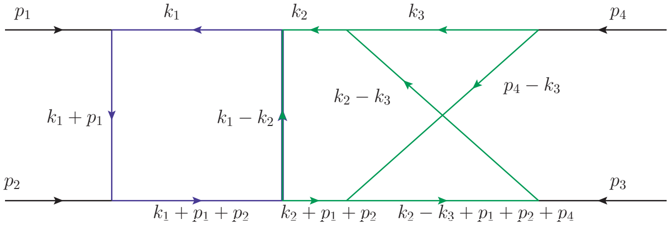
Final MBrepresentation (6-dim):

```
((-s)^z1 (-t)^(-3 eps-z1-z2) (-u)^z2 Gamma[-z1] Gamma[-z2]
Gamma[3 eps+z1+z2] Gamma[-z3] Gamma[-eps-z2-z4-z5]
Gamma[-z5] Gamma[1-3 eps-z1+z5] Gamma[1- eps+z2+z4+z5]
Gamma[1-3 eps-z1+z3+z4+z5] Gamma[2 eps+z1+z2+z4-z6]
Gamma[-1+4 eps+z1+z2-z3-z4-z5-z6] Gamma[-z6]
Gamma[1-2 eps+z6] Gamma[1-3 eps-z2+z3+z6]
Gamma[1-3 eps-z2-z4+z6] Gamma[1-4 eps-z1-z2+z3+z5+z6])
/(Gamma[2-4 eps] Gamma[1-eps+z2+z4+z5-z6]
Gamma[1-3 eps-z2-z4-z5+z6] Gamma[2-6 eps-z1-z2+z3+z5+z6])
```

Numerical crosscheck $s = t = u = -1$:

```
AMBRE+MB:  {26.5404 + 2.40412/eps, {0.00580197 + 2.8415*10^-6/eps
FIESTA:    {26.5387 + 2.40417/eps, {0.00021977 + 0.0000206936/eps
```

Hybrid method for 3–loops



Hybrid method: LA– $\{k_1\}$; GA– $\{k_2, k_3\}$

Towards analytical solutions

— The Mathematica package MBsums (package by M. Ochman) transforms MB integrals into sums by Cauchy theorem
— The current version of MBsums is 1.0, other packages: Sigma by C. Schneider, XSummer by S. Moch and P. Uwer
— An example. Let

```
int = MBint[-((-x)^(z1 + z6)*y^z2*Gamma[-z1]*
      Gamma[1 + z1]*Gamma[-z1 - z2]*Gamma[-z2]*
      Gamma[z2]*Gamma[-z1 + z2]*Gamma[-z2 - z6]*
      Gamma[z2 - z6]*Gamma[-z6]*Gamma[1 + z6])/
      (2*eps*Gamma[-2*z1]*Gamma[1 - z2]*
      Gamma[1 + z2]*Gamma[-2*z6]), {{eps -> 0},
      {z1 -> -1/2, z2 -> -11/192, z6 -> -1/2}}]
```

There is a lot of space for activity here!

Then

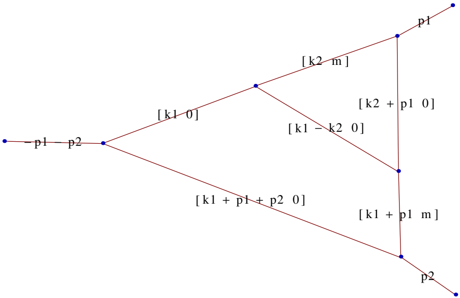
```
MBIntToSum[int, {x -> -5, y -> 7},
                {z1 -> L, z6 -> L, z2 -> L}]
```

gives

```
{MBsum[-((-x)^(-n1 - n2)*(-1 + n3)!^2*(n1 + n3)!*
      (n2 + n3)!*(2*HarmonicNumber[-1 + n3] -
      2*HarmonicNumber[n3] - HarmonicNumber[
       -1 - n1 + n3] + HarmonicNumber[n1 + n3] -
      HarmonicNumber[-1 - n2 + n3] +
      HarmonicNumber[n2 + n3] - Log[y]))/
      (2*(-1)^(2*n3)*eps*x^2*y^n3*(1 + 2*n1)!*
      (1 + 2*n2)!*n3!^2*(-1 - n1 + n3)!*
      (-1 - n2 + n3)!), n1 >= 0 && n2 >= 0 &&
      n3 >= 1 && 1 + n1 <= n3 && 1 + n2 <= n3,
      {n1, n2, n3}],MBsum[...],...}
```

Numerical results

Example, $((-\mathbf{1})^{(-\mathbf{z1})})$!



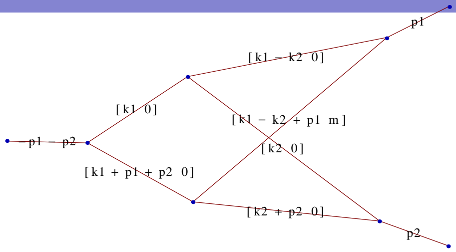

```
{MBint[((-1)^(-z1) Gamma[-1-z1] Gamma[2+z1] Gamma[-1-z2]
 Gamma[z1-z2] Gamma[1+z2-z3]^2 Gamma[-z3] Gamma[1+z3]
Gamma[-z1+z3]^2 Gamma[-z2+z3])/(Gamma[-z1] Gamma[1+z1-z2]
           Gamma[1-z1+z3]), {{eps -> 0},
    {z1 -> -47/37, z2 -> -139/94, z3 -> -176/235}}]}
```

Last improvements towards stable results (MBnumerics, J. Usovitsch, E. Dubovyk):

Analytical:	-1.199526183135566 + 5.567365907880696 I
Our MBnumerics:	-1.199526183168498 + 5.567365907904922 I
MB(Vegas):	-1.199561086311856 + 5.569395048002913 I
MB(Cuhre):	NaN
FIESTA:	-1.200370278497323 + 5.561435923863947 I
SecDec:	no output

"Euclidean" results:

Analytical result:	3.376807975550548
MB(Vegas):	3.376922163980158
MB(Cuhre):	3.376807975447292
FIESTA:	3.376815834907247
SecDec:	big error



Results (constant part of the MBintegrals):

```
Analytical:       -0.7785996089679684 - 4.123512593396311 I
Our MBnumerics:   -0.778599608324769 - 4.123512600516016 I
MB(Vegas):                              big error
MB(Cuhre):                                 NaN
FIESTA:                                 big error
SecDec:                                 no output
```

Euclidean results (constant part):

```
Analytical:         -0.4966198306057021
MB(Vegas):          -0.4969417442183914
MB(Cuhre):          -0.4966198313219404
FIESTA:             -0.4966184488196595
SecDec:             -0.4966192150541896
```

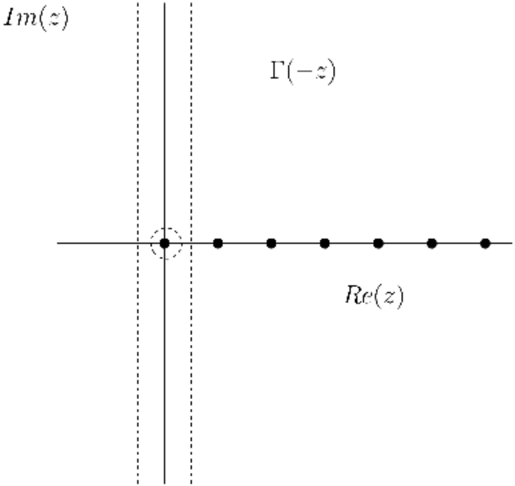
Extracting Singularities

MB method is very useful for analazing singularities of Feynman integrals, in most cases we can get them just in analytical form or can be sum up easily

Expansion in ϵ using MB.m by M. Czakon

What does it mean "Make analytic continuation"? - first idea by B. Tausk, later by Smirnov & Smirnov

Extracting Singularities $\epsilon \to 0$



```
(* shifting contours *)

:=
sim = Gamma[-z]
]=
Gamma[-z]

:=
Sum[-Residue[Gamma[-z], {z, n}], {n, 0, 100}] // N
]=
0.367879

:=
n1 = NIntegrate[
   1 / (2 Pi) sim /. z → -1 / 20 + I y, {y, -10, 10}]
]=
0.367879 + 0. i

:=
n2 = NIntegrate[
   1 / (2 Pi) sim /. z → 1 / 20 + I y, {y, -10, 10}]
]=
-0.632121 + 0. i

:=
n2 - n1
]=
-1. + 0. i

:=
Residue[sim, {z, 0}]
]=
-1
```

n2=n1+Residue[sim,{z,0}]

Hands-on examples:
`ReIm_oscillations.nb`, `SE2l2m.nb`, `B5l2m.nb`
`B5nf_0external.nb` (What is the leading singularity in ϵ?)
-analytical solutions by matchings
-approximations
-MB integrations in Minkowskian

Summary

- a construction of MB integrals is optimized well up to two-loops, more complicated at three-loops
- usually high dimensional integrals for variety of masses, legs, loops involved,
- solving analytically MB integrals through nested sums - hard thing - place for many improvements and new ideas
- numerical approach very promising, first package already available internally which allows for physical applications beyond present capabilities - more at LL 2016 and forthocoming publication with DESY Tord Riemann's group

Backup material

Alpha and Feynman Parameters

Feynman parameters representation:

$$\frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}} = \frac{\Gamma(n_1 + \dots + n_N)}{\Gamma(n_1) \dots \Gamma(n_N)} \int\limits_0^1 dx_1 \dots \int\limits_0^1 dx_N \frac{x_1^{n_1 - 1} \dots x_N^{n_N - 1} \delta(1 - x_1 - \dots - x_m)}{(x_1 D_1 + \dots + x_N D_N)^{N_\nu}}$$

with $N_\nu = n_1 + \dots + n_N$.
Alpha parameters representation:

$$\frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}} = \frac{i^{-N_\nu}}{\Gamma(n_1) \dots \Gamma(n_N)} \int\limits_0^\infty d\alpha_1 \dots \int\limits_0^\infty d\alpha_N \alpha_1^{n_1 - 1} \dots \alpha_N^{n_N - 1} e^{i[\alpha_1 D_1 + \dots + \alpha_N D_N]}$$

Using the identity

$$1 = \int\limits_0^\infty \frac{d\lambda}{\lambda} \delta \left(1 - \frac{1}{\lambda} \sum_{i=1}^{N} \alpha_i \right)$$

and change variables from α_i to $\alpha_i = \lambda x_i$, one can find

$$\frac{1}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} = \frac{i^{-N_\nu}}{\Gamma(n_1)\ldots\Gamma(n_N)} \int_0^\infty dx_1 \ldots \int_0^\infty dx_N \, x_1^{n_1-1} \ldots x_N^{n_N-1} \times \int_0^\infty d\lambda \lambda^{N_\nu -1} \delta\left(1 - \sum_{i=1}^N x_i\right) e^{i\lambda \sum_{i=1}^N x_i D_i}.$$

Integrating over λ we come to Feynman parameters representation

$$\frac{1}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} = \frac{\Gamma(n_1+\ldots+n_N)}{\Gamma(n_1)\ldots\Gamma(n_N)} \int_0^\infty dx_1 \ldots \int_0^\infty dx_N \frac{x_1^{n_1-1} \ldots x_N^{n_N-1} \delta(1 - \sum_{i=1}^N x_i)}{\left(\sum_{i=1}^N x_i D_i\right)^{N_\nu}}$$

Note that all x_i are positive while the sum of x_i must be unity. Therefore the integration region can be limited to $0 < x_i < 1$

$$0 < x_i < 1 \;\; \Leftrightarrow \;\; 0 < x_i < \infty$$

Let now consider the momentum dependent function

$$m^2 = x_1 D_1 + \ldots + x_i D_i + \ldots + x_N D_N = k_i M_{ij} k_j - 2Q_j k_j + J$$

with M – $(L \times L)$-matrix, $Q = Q(x_i, p_e)$ – L-vector and $J = J(x_i x_j, m_i^2, p_{e_i} p_{e_j})$.
Integration over loop momenta:

- Shift momenta to remove linear terms in k

$$k \to k + M^{-1}Q \Rightarrow m^2 = kMk - QM^{-1}Q + J$$

  (shifts leave integral unchanged)

- Wick rotation – transforms Minkowski space into an Euclidean for each loop momenta

$$k_0 \to ik_0;\;\; k_j \to k_j (1 \leqslant j \leqslant d-1) \Rightarrow k^2 \to -k^2;\;\; d^d k \to i d^d k$$

- diagonalization of the matrix M

$$k^\dagger M k = (V(x)k)^\dagger V(x) M V(x)^{-1} V(x) k;\;\; k(x) = V(x)k;\;\; VMV^{-1} = M_{\text{diag}};\;\; (V^\dagger = V^{-1})$$

$$kMk \Rightarrow k(x) M_{\text{diag}} k(x) = \sum_{i=1}^{L} \alpha_i k_i^2(x)$$

  (leaves integral unchanged)

After such manipulation function m^2 has following form

$$m^2 = -\sum_{i=1}^{L} \alpha_i k_i^2 - QM^{-1}Q + J$$

- Now we rescale k_i

$$k_i \to \sqrt{\alpha_i} k_i \Rightarrow d^d k_i \to (\alpha_i)^{-d/2} d^d k_i \quad \text{and} \quad \prod_{i=1}^{L} \alpha_i = detM$$

Finally, we obtaine

$$G(X) = (-1)^{N_\nu} (i)^L (detM)^{-d/2} \frac{\Gamma(N_\nu)}{\prod_{i=1}^{N} \Gamma(n_i)} \int dx_1 \dots dx_N \int \frac{Dk_1 \dots Dk_L}{\left(\sum_{i=1}^{L} k_i^2 + QM^{-1}Q - J\right)^{N_\nu}}$$

or

$$G(X) = \frac{(i)^{L-N_\nu} (detM)^{-d/2}}{\prod_{i=1}^{N} \Gamma(n_i)} \int d\alpha_1 \dots d\alpha_N \int Dk_1 \dots Dk_L \; e^{-i\left(\sum_{i=1}^{L} k_i^2 + QM^{-1}Q - J\right)}$$

with $Dk = \dfrac{d^d k}{i\pi^{d/2}}$.

Now integration over loop momenta can be done in the simple way

$$i^L \int \frac{Dk_1 \dots Dk_L}{\left(\sum\limits_{i=1}^{L} k_i^2 + \mu^2(x)\right)^{N_\nu}} = \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(N_\nu)} \frac{1}{(\mu^2(x))^{N_\nu - \frac{dL}{2}}}$$

$$\int Dk_1 \dots Dk_L e^{-i\left(\sum\limits_{i=1}^{L} k_i^2 + \mu^2(\alpha)\right)} = (-i)^{-Ld/2} e^{-i\mu^2(\alpha)}$$

with $\mu^2(x) = QM^{-1}Q - J$. Final result is

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod\limits_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j \, x_j^{n_j-1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

$$G(X) = \frac{(i)^{L-N_\nu}(detM)^{-d/2}}{\prod\limits_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} d\alpha_j \, \alpha_j^{n_j-1} e^{-i\frac{F(\alpha)}{U(\alpha)}}$$

where we introduce two Feynmann graph polynomials U and F

$$U = det\ M, \quad F = -det\ M\ J + QM^T Q \Leftrightarrow m^2 = kMk - 2Qk + J$$

MB Representation

MB relation in the general case

$$\frac{1}{(A_1+\ldots+A_n)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{(2\pi i)^{n-1}}\int_{-i\infty}^{i\infty} dz_1\ldots dz_{n-1} \times \prod_{i=1}^{n-1} A_i^{z_i}\, A_n^{-\lambda-z_1-\ldots-z_{n-1}} \prod_{i=1}^{n-1}\Gamma(-z_i)\,\Gamma(\lambda+z_1+\ldots+z_{n-1})$$

Integration over $\{x_1,\ldots,x_N\}$

$$\int_0^1 \prod_{i=1}^{N} dx_i\, x_i^{n_i-1}\,\delta(1-x_1-\ldots-x_N) = \frac{\Gamma(n_1)\ldots\Gamma(n_N)}{\Gamma\left(n_1+\ldots+n_N\right)}$$

Implement Barnes lemmas to improve dimensionality

$$\int_{-i\infty}^{i\infty} dz\Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(d-z) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

$$\int_{-i\infty}^{i\infty} dz\frac{\Gamma(a+z)\Gamma(b+z)\Gamma(c+z)\Gamma(d-z)\Gamma(e-z)}{\Gamma(a+b+c+d+e+z)} = \frac{\Gamma(a+d)\Gamma(a+e)\Gamma(b+d)\Gamma(b+e)\Gamma(c+d)\Gamma(c+e)}{\Gamma(a+b+d+e)\Gamma(a+c+d+e)\Gamma(b+c+d+e)}.$$

Cheng–Wu Theorem

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j\, x_j^{n_j-1} \delta(1-\sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

The Cheng–Wu theorem states that the same formula holds with the delta function

$$\delta\left(\sum_{i\in\Omega} x_i - 1\right)$$

where Ω is an arbitrary subset of the lines $1,\dots,L$, when the integration over the rest of the variables, i.e. for $i \notin \Omega$, is extended to the integration from zero to infinity.
One can prove this theorem in a simple way starting from the alpha representation using

$$1 = \int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1 - \frac{1}{\lambda}\sum_{i=1}^{N}\alpha_i\right) \Leftrightarrow 1 = \int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1 - \frac{1}{\lambda}\sum_{i\in\Omega}\alpha_i\right)$$

and change variables from α_i to $\alpha_i = \lambda x_i$ as shown above.

Non–Planar DoubleBox

J. B. Tausk,
"Nonplanar massless two loop Feynman diagrams with four on-shell legs",
Phys. Lett. B **469** (1999) 225; [hep-ph/9909506]

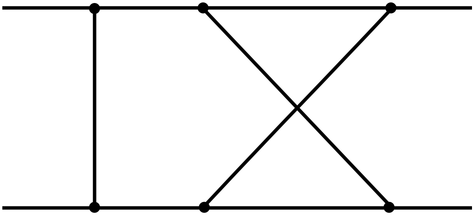


Figure 5: The non-planar double box.

$$B_7^{NP} = \iint d^dk_1 d^dk_2 \frac{1}{[(k_1+k_2+p_1+p_2)^2]^{n_1}[(k_1+k_2+p_2)^2]^{n_2}[(k_1+K_2)^2]^{n_3}} \frac{1}{[(k_1-p_3)^2]^{n_4}[(k_1)^2]^{n_5}[(k_2-p_4)^2]^{n_6}[(k_2)^2]^{n_7}}$$

$$U(x) = x[1]x[2] + x[1]x[4] + x[2]x[4] + x[1]x[5] + x[2]x[5] + x[2]x[6] + x[4]x[6] + x[5]x[6] + x[1]x[7] + x[4]x[7] + x[5]x[7] + x[6]x[7]$$

$$F(x) = -s\, x[1]x[2]x[5] - s\, x[1]x[3]x[5] - s\, x[2]x[3]x[5] - u\, x[2]x[4]x[6] - s\, x[3]x[5]x[6] - t\, x[1]x[4]x[7] - s\, x[3]x[5]x[7] - s\, x[3]x[6]x[7]$$

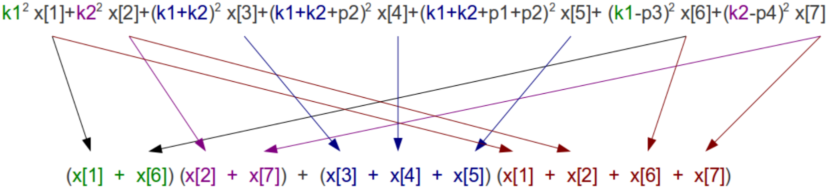


Figure 6: Factorization scheme

$$U(x) = (x[1]+x[6])(x[2]+x[7]) + (x[3]+x[4]+x[5])(x[1]+x[2]+x[6]+x[7])$$

$$F(x) = -t\, x[1]x[4]x[7] - u\, x[2]x[4]x[6] - s\, x[1]x[2]x[5] - s\, x[3]x[6]x[7] - s\, x[3]x[5](x[1]+x[2]+x[6]+x[7])$$

Now we can apply Cheng-Wu theorem and integration will look as follows

$$B_7^{NP} = \frac{(-1)^{N_\nu}\Gamma(N_\nu - d)}{\Gamma(n_1)\dots\Gamma(n_N)} \int_0^\infty dx_3 dx_4 dx_5 \int_0^1 dx_3 dx_4 dx_5 \delta(1-(x_1+x_2+x_6+x_7)) \frac{((x_1+x_6)(x_2+x_7)+x_3+x_4+x_5)^{N_\nu - \frac{3d}{2}}}{(-t\, x_1x_4x_7 - u\, x_2x_4x_6 - s\, x_1x_2x_5 - s\, x_3x_6x_7 - s\, x_3x_5)^{N_\nu - d}}$$

$$B_7^{NP} = \frac{(-1)^{N_\nu}}{\Gamma(n_1)\dots\Gamma(n_N)} \int_{-i\infty}^{i\infty} dz_1 \dots dz_2 \int dx_1 \dots dx_7\, (-s)^{-N_\nu+d-z2-z3}(-t)^{z2}(-u)^{z3} \times \Gamma(-z1)\Gamma(-z2)\Gamma(-z3)\Gamma(-z4)\Gamma(N_\nu - d + z1 + z2 + z3 + z4) \times x_1^{-N_\nu+d-z1-z2-z3} x_2^{z2+z3} x_3{}^{-N_\nu+d-z2-z3-z4} x_4{}^{z1+z3} x_5{}^{z2+z4} x_6^{z1+z2} x_7^{z3+z4} \times (x_3+x_4+x_5+(x_1+x_6)(x_2+x_7))^{N_\nu - \frac{3d}{2}}$$

Integration over Cheng–Wu variables

$$\int_0^\infty dx\, x^{N_1}(x+A)^{N_2} = \frac{A^{1+N_1+N_2}\Gamma(1+N_1)\Gamma(-1-N_1-N_2)}{\Gamma(-N_2)}$$

4-dim result:

$$B_7^{NP} = \frac{(-1)^{N_\nu}}{\Gamma(n_1)\ldots\Gamma(n_7)} \int_{-i\infty}^{i\infty} dz_1 \ldots dz_4 (-s)^{4-2\epsilon-N_\nu-z_{23}}(-t)^{z_3}(-u)^{z_2}$$
$$\frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(2-\epsilon-n_{45})\Gamma(2-\epsilon-n_{67})}{\Gamma(4-2\epsilon-n_{4567})\Gamma(n_{45}+z_{1234})\Gamma(n_{67}+z_{1234})\Gamma(6-3\epsilon-N_\nu)}$$
$$\Gamma(n_2+z_{23})\Gamma(n_4+z_{24})\Gamma(n_5+z_{13})\Gamma(n_6+z_{34})\Gamma(n_7+z_{12})\Gamma^3(-2+\epsilon+n_{4567}+z_{1234})$$
$$\Gamma(4-2\epsilon-n_{124567}-z_{123})\Gamma(4-2\epsilon-n_{234567}-z_{234})\Gamma(-4+2\epsilon+N_\nu+z_{1234})$$

with notations $z_{i\ldots j\ldots k} = z_i + \ldots + z_j + \ldots + z_k$
and $n_{i\ldots j\ldots k} = n_i + \ldots + n_j + \ldots + n_k$

Non–Planar Vertex

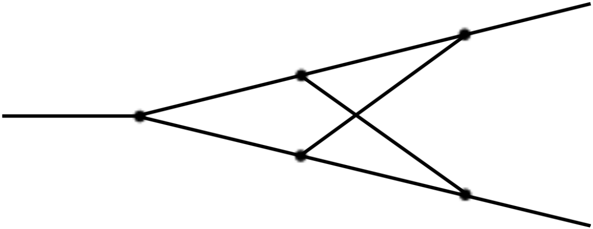


Figure 7: The non-planar vertex.

$$V_6^{NP} = \iint d^d k_1 d^d k_2 \frac{1}{[k_1^2]^{n_1}[(p_1-k_1)^2]^{n_2}[(p_1-k_1-k_2)^2]^{n_3}}$$
$$\frac{1}{[(p_2+k_1+k_2)^2]^{n_4}[(p_2+k_2)^2]^{n_5}[k_2^2]^{n_6}}$$

$$\begin{aligned} m^2 = \sum x_i D_i \quad &= \quad x_1(p_1 - k_1 - k_2)^2 & x_1 &\to v_1 C_1 \\ &+ \quad x_2(p_2 + k_1 + k_2)^2 & x_2 &\to v_1 C_2 \\ &+ \quad x_3(k_1)^2 & x_3 &\to v_2 A_1 \\ &+ \quad x_4(p_1 - k_1)^2 & x_4 &\to v_2 A_2 \\ &+ \quad x_5(p_2 + k_2)^2 & x_5 &\to v_3 B_1 \\ &+ \quad x_6(k_2)^2 & x_6 &\to v_3 B_2 \end{aligned}$$

$$\delta\left(1 - \sum_{i=1}^{6} x_i\right) \Rightarrow \delta(1 - v_1 - v_2 - v_3)\delta(1 - A_1 - A_2)\delta(1 - B_1 - B_2)\delta(1 - C_1 - C_2)$$

Jacobian of the transformation:

$$J = v_1^{N_C - 1} v_2^{N_A - 1} v_3^{N_B - 1} = v_1 v_2 v_3$$

- Using $\delta(1 - A_1 - A_2)\delta(1 - B_1 - B_2)\delta(1 - C_1 - C_2)$ we can simplify U and F

$$U = v_1 v_2 + v_1 v_3 + v_2 v_3 \quad F = -sA_1 B_2 C_1 v_1 v_2 v_3 - sA_2 B_1 C_2 v_1 v_2 v_3 - sC_1 C_2 v_1 v_3^2 - sC_1 C_2 v_2 v_3^2$$

- Choose now v_3 as Cheng-Wu variable $\int_0^\infty dv_3 \int_0^1 dv_1 dv_2 \delta(1 - v_1 - v_2)$

$$U = v_3 + v_1 v_2 \quad F = -sA_1B_2C_1v_1v_2v_3 - sA_2B_1C_2v_1v_2v_3 - sC_1C_2v_1v_3^2$$

- Apply MB relation for F

$$V_6^{NP} = (-1)^{N_\nu} \int_{-i\infty}^{i\infty} dz_1 dz_2 \int_0^\infty dv_3 \int_0^1 dv_1 dv_2 dA_1 \ldots dC_2 \, (-s)^{4-2\epsilon-N_\nu}$$
$$A_1^{-1+n_1+z_1} A_2^{-1+n_2+z_2} B_1^{-1+n_6+z_2} B_2^{-1+n_5+z_1} C_1^{3-2\epsilon-n_{12456}-z_2} C_2^{3-2\epsilon-n_{12356}-z_1}$$
$$v_1^{-1+n_{12}+z_{12}} v_2^{-1+n_{56}+z_{12}} v_3^{7-4\epsilon-2n_{1256}-n_{34}-z_{12}} (v_1v_2 + v_3)^{-6+3\epsilon+N_\nu}$$
$$\frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma(-4+2\epsilon+N_\nu+z_{12})}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)\Gamma(n_4)\Gamma(n_5)\Gamma(n_6)}$$

- Iintegrate over $v3$ using

$$\int_0^\infty dx\, x^{N_1}(x+A)^{N_2} = \frac{A^{1+N_1+N_2}\Gamma(1+N_1)\Gamma(-1-N_1-N_2)}{\Gamma(-N_2)}$$

$$V_6^{NP} = (-1)^{N_\nu} \int_{-i\infty}^{i\infty} dz_1 dz_2 \int_0^1 dv_1 dv_2 dA_1 \ldots dC_2 (-s)^{4-2\epsilon-N_\nu}$$
$$A_1^{-1+n_1+z_1} A_2^{-1+n_2+z_2} B_1^{-1+n_6+z_2} B_2^{-1+n_5+z_1} C_1^{3-2\epsilon-n_{12456}-z_2} C_2^{3-2\epsilon-n_{12356}-z_1}$$
$$v_1^{1-eps-n_{56}} v_2^{1-eps-n_{12}} \Gamma(8-4\epsilon-2n_{1256}-n_{34}-z_{12})$$
$$\frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma(-4+2\epsilon+N_\nu+z_{12})\Gamma(-2+\epsilon+n_{1256}+z_{12})}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)\Gamma(n_4)\Gamma(n_5)\Gamma(n_6)\Gamma(6-3\epsilon-N_\nu)}$$

- Iintegrate over each subset of variables $\{v, A, B, C\}$ separately using

$$\int_0^1 \prod_{i=1}^{N} dx_i \; x_i^{n_i-1} \; \delta(1 - x_1 - \ldots - x_N) = \frac{\Gamma(n_1)\ldots\Gamma(n_N)}{\Gamma\left(n_1 + \ldots + n_N\right)}$$

and get 2-dim representation:

$$V_6^{NP} = (-1)^{N_\nu} (-s)^{4-2\epsilon-N_\nu} \int_{-i\infty}^{i\infty} dz_1 dz_2$$

$$\frac{\Gamma(2-eps-n_{12})\Gamma(2-eps-n_{56})\Gamma(4-2eps-n_{12356}-z_1)\Gamma(-z_1)\Gamma(n_1+z_1)}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)\Gamma(n_4)\Gamma(n_5)\Gamma(4-2eps-n_{1256})\Gamma(n_6)\Gamma(n_5+z_1)}$$

$$\frac{\Gamma(-z_2)\Gamma(n_2+z_2)\Gamma(n_6+z_2)\Gamma(-4+2eps+N_\nu+z_{12})}{\Gamma(8-4eps-2n_{1256}-n_{34}-z_{12})\Gamma(n_{12}+z_{12})\Gamma(n_{56}+z_{12})\Gamma(4-2eps-n_{12456}-z_2)}$$

MB.m, integration in Euclidean region

Steps for numerical integration:

- real parametrization $z_i \to c_i + It_i, \quad t_i \in (-\infty, \infty)$
- **MB.m way:** transformation to finite integration interval (here $[0,1]$ and linked with CUBA library)

$$t_i \to Log\left[\frac{x_i}{1-x_i}\right] \quad dt_i \to \frac{dx_i}{x_i(1-x_i)}$$

In general, the factor $(-1)^{-z1} \to e^{-\pi t_1}$
may lead to problems in asymptotic limit $t_1 \to -\infty$.
Fortunately, this factor cancels with remaining gammas, taking

$$\lim_{t\to\pm\infty} \Gamma(a+It) \sim e^{-\frac{\pi|t|}{2}} t^{a-\frac{1}{2}}$$

and a ray $t_1 = t, \ t_2 = 0, \ t_3 = 0$ we can compute a limit for a product of gamma functions:

$$\frac{\prod_j \mathbf{G_j}(N_j)}{\prod_k \mathbf{G_k}(N_k)} \sim e^{\pi t} \frac{1}{t^{646/235}}$$

What remains is $\sim \dfrac{1}{t^\alpha}$.

However, transformation to the finite region does not remove singular behavior of the integrand

$$\frac{1}{x\,Log[x]^{646/235}} \xrightarrow{x\to 0} \infty$$

This kind of sigularity is integratable when $\alpha > 1$ (our example)
The solution in general is to take **another transformation**

$$t_i \to Tan[\pi(x_i - \frac{1}{2})] \quad dt_i \to \frac{\pi dx_i}{Cos[\pi(x_i - \frac{1}{2})]^2}$$

Now we have

$$\frac{Cos[\pi(x_i - \frac{1}{2})]^{176/235}}{Sin[\pi(x_i - \frac{1}{2})]^{646/235}} \xrightarrow{x\to 0} 0$$