Non-planar diagrams and Mellin-Barnes representations

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Outline

Introduction

L–loop n–point functions

Consider an arbitrary L–loop integral $G(X)$ with loop momenta k_l , with E external legs with momenta p_e and with N internal lines with masses m_i and propagators $1/D_i$

$$G(X) = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \ldots d^d k_L \, X(k_1, \ldots, k_L)}{D_1^{n_1} \ldots D_i^{n_i} \ldots D_N^{n_N}}$$

$$d = 4 - 2\epsilon$$

$$D_i = q_i^2 - m_i^2 = \left[\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{M} d_i^e p_e \right] - m_i^2$$

$X(k_1, \ldots, k_L)$ stands for tensors in the loop momenta.

Two representations for integrals

Feynman parameter representation ($N_\nu = n_1 + \ldots + n_N$):

$$\frac{1}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} = \frac{\Gamma(n_1 + \ldots + n_N)}{\Gamma(n_1) \ldots \Gamma(n_N)} \int_0^1 dx_1 \ldots \int_0^1 dx_N \frac{x_1^{n_1 - 1} \ldots x_N^{n_N - 1} \delta(1 - x_1 - \ldots - x_m)}{(x_1 D_1 + \ldots + x_N D_N)^{N_\nu}}$$

Alpha parameter representation:

$$\frac{1}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} = \frac{i^{-N_\nu}}{\Gamma(n_1) \ldots \Gamma(n_N)} \int_0^\infty d\alpha_1 \ldots \int_0^\infty d\alpha_N \alpha_1^{n_1 - 1} \ldots \alpha_N^{n_N - 1} e^{i[\alpha_1 D_1 + \ldots + \alpha_N D_N]}$$

Using the identity

$$1 = \int_0^\infty \frac{d\lambda}{\lambda} \delta\left(1 - \frac{1}{\lambda} \sum_{i=1}^N \alpha_i\right)$$

and by change of variables from α_i to $\alpha_i = \lambda x_i$, one can find

$$\frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}} = \frac{i^{-N_\nu}}{\Gamma(n_1)\dots\Gamma(n_N)} \int_0^\infty dx_1 \dots \int_0^\infty dx_N \, x_1^{n_1-1} \dots x_N^{n_N-1}$$
$$\times \int_0^\infty d\lambda \lambda^{N_\nu - 1} \delta\left(1 - \sum_{i=1}^N x_i\right) e^{i\lambda \sum\limits_{i=1}^N x_i D_i}.$$

Integrating over λ we come to the Feynman parameter representation

$$\frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}} = \frac{\Gamma(n_1 + \dots + n_N)}{\Gamma(n_1)\dots\Gamma(n_N)} \int_0^\infty dx_1 \dots \int_0^\infty dx_N \frac{x_1^{n_1-1} \dots x_N^{n_N-1} \delta(1 - \sum\limits_{i=1}^N x_i)}{\left(\sum\limits_{i=1}^N x_i D_i\right)^{N_\nu}}$$

Note that all x_i are positive while the sum of x_i must be unity. Therefore the integration region can be limited to $0 < x_i < 1$

$$0 < x_i < 1 \quad \Leftrightarrow \quad 0 < x_i < \infty$$

- Both representations have been used heavily since the beginning of one-loop calculations.
- There are also other methods of Feynman integral calculations, e.g. differential equations (which got lately a new strong kick initiated by J. Henn) [1, 2, 3], expansions by regions [4], sector decomposition [5, 6, 7, 8, 9, 10, 11, 12], etc.
- However, one method is still in our opinion not exploited sufficiently, namely solving Feynman integrals by Mellin-Barnes representations [13, 14, 15, 16, 17, 18]. There is steady progress in its automatization (see [19, 20, 21, 22, 23, 24], with a software available at http://projects.hepforge.org/mbtools, but the method is less developed so far than, e.g., that of differential equations.

It would be wonderful to have an algorithm for automatic evaluation of all the scalar (and tensor) integrals by infinite sums!

For not too involved classes of functions, see Summer, at http://www.nikhef.nl/ t68/ and XSUMMER [26].
For automation we need to

1. construct MB representations this talk: non-planar case (I. Dubovyk et al., unpubl.)
2. change them into nested sums this talk: MBtoSums package (M. Ochman et al., unpubl.)
3. solve analytically this talk and the talk by Clemens Raab

$$Certainly,\ there\ are\ limitations:$$

- Number of loops: One-loop, two-loop,...?
- Number of scales: Massive, off-shell?
- Number of legs: 2-,3-,.... point functions?

Starting point

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j \, x_j^{n_j-1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

The functions U and F are called graph or Symanzik polynomials.

General MB relation can be applied to polynomials U and F

$$\frac{1}{(A_1 + \ldots + A_n)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{(2\pi i)^{n-1}} \int_{-i\infty}^{i\infty} dz_1 \ldots dz_{n-1}$$
$$\times \prod_{i=1}^{n-1} A_i^{z_i} \, A_n^{-\lambda - z_1 - \ldots - z_{n-1}} \prod_{i=1}^{n-1} \Gamma(-z_i) \, \Gamma(\lambda + z_1 + \ldots + z_{n-1})$$

Then we get multidimensional `MB` integrals.

MB integrals and the iterative loop–by–loop (LA) approach

Examples, description, links to basic tools and literature:
http://us.edu.pl/∼ gluza/ambre/

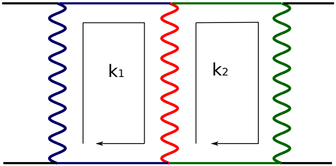


Figure: Loop–by–loop (LA) example

Here: $U(x) \equiv 1$
Input:

$$\mathrm{PR}[\mathrm{k1}, \mathrm{m}, \mathrm{n1}]\mathrm{PR}[\mathrm{k1}+\mathrm{p1}, 0, \mathrm{n2}]\mathrm{PR}[\mathrm{k1}+\mathrm{p1}+\mathrm{p2}, \mathrm{m}, \mathrm{n3}]\mathrm{PR}[\mathrm{k1}-\mathrm{k2}, 0, \mathrm{n4}]$$
$$\mathrm{PR}[\mathrm{k2}, \mathrm{m}, \mathrm{n5}]\mathrm{PR}[\mathrm{k2}+\mathrm{p1}+\mathrm{p2}, \mathrm{m}, \mathrm{n6}]\mathrm{PR}[\mathrm{k2}-\mathrm{p3}, 0, \mathrm{n7}]$$

Integration over k_2:

$$\mathrm{PR[k1-k2,0,n4]PR[k2,m,n5]PR[k2+p1+p2,m,n6]PR[k2-p3,0,n7]}$$

$$\mathrm{F[X] = m^2\,(X[2]+X[3])^2 - PR[k1,m]X[1]X[2] - PR[k1+p1+p2,m]X[1]X[3]}$$
$$\mathrm{-sX[2]X[3] - PR[k1-p3,0]X[1]X[4]}$$

Integration over k_1:

$$\mathrm{PR[k1,m,\alpha]PR[k1+p1,0,n2]PR[k1+p1+p2,m,\beta]PR[k1-p3,0,\gamma]}$$

$$\mathrm{F[X] = m^2\,(X[1]+X[3])^2 - sX[1]X[3] - tX[2]X[4]}$$

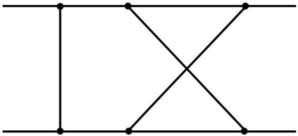
Dimensions of ladder planar MB integrals	Massless cases				Massive cases			
Number of loops (L)	1	2	3	4	1	2	3	4
No Barnes First Lemma	1	4	7	10	3	8	13	18
With BFL	1	4	7	10	2 (1+1)	6 (4+2)	10 (7+3)	14 (10+4)

Optimal results:

$$\mathrm{Dim(massive) = Dim(massless) + \#loops}$$

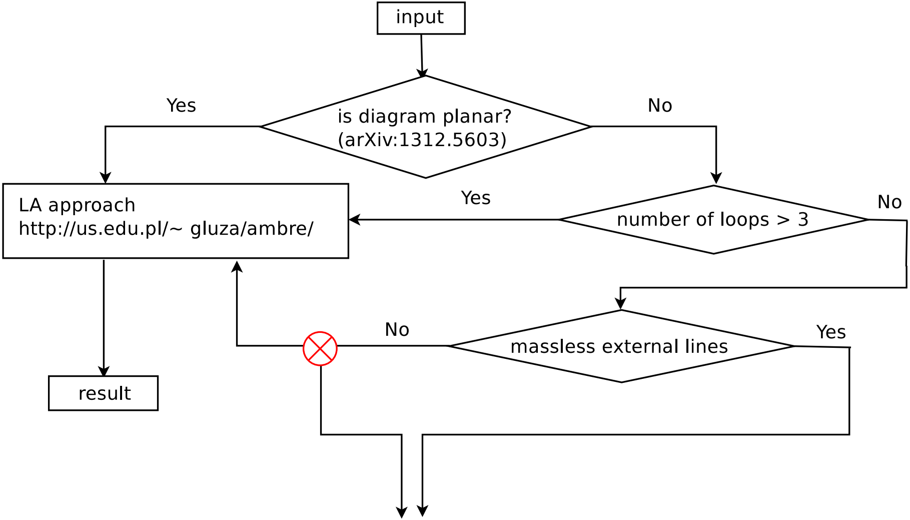
Global approach - GA

Sometimes it is better to change into the MB representation the complete U and F polynomials,
e.g.

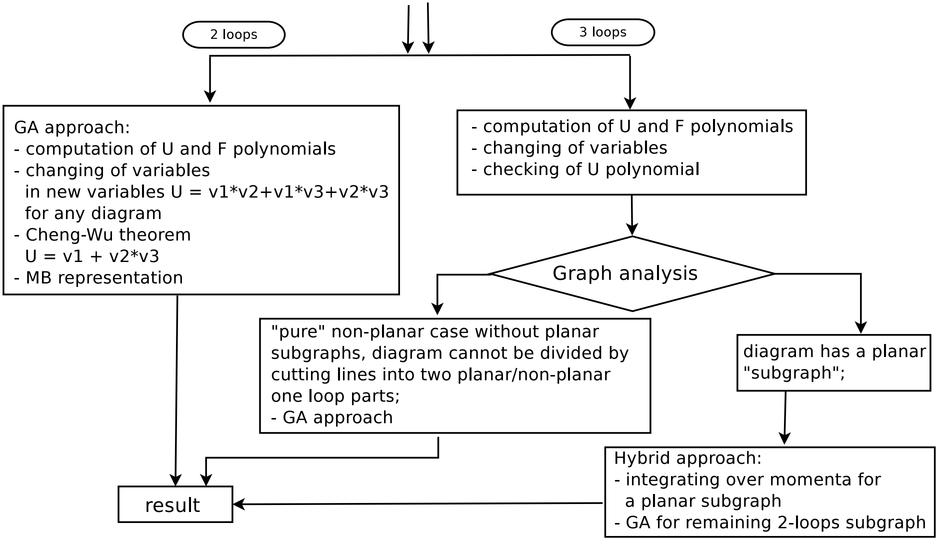


1. massless case: 4-dim MB - GA
2. massive case: 8-dim MB - LA (with GA not less than 10-dim MB (Heinrich, Smirnov, PLB 2004)

1. AMBRE reloaded - non-planar version, basic chart (I)



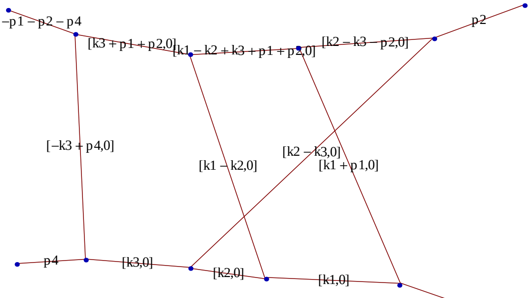
Basic chart (II), GA



```
<< PlanarityTest_v1.1.m
by E. Dubovyk and K. Bielas ver: 1.1
created: January 2014
last executed: 25.04.2014 at 16:48

PlanarityTest[{PR[k1, 0, n1] PR[k1 + p1, 0, n2] ...
               PR[k3, 0, n10] PR[...]}, {k1, k2, k3},
               DrawGraph -> True];
```

Message displayed: The Diagram is non-planar.



Cheng–Wu Theorem

$$G(X)=\frac{(-1)^{N_\nu}\Gamma\left(N_\nu-\frac{d}{2}L\right)}{\prod_{i=1}^{N}\Gamma(n_i)}\int\prod_{j=1}^{N}dx_j\,x_j^{n_j-1}\delta(1-\sum_{i=1}^{N}x_i)\frac{U(x)^{N_\nu-d(L+1)/2}}{F(x)^{N_\nu-dL/2}}$$

The Cheng–Wu theorem [25] states that the same formula holds with the delta function

$$\delta\left(\sum_{i\in\Omega}x_i-1\right)$$

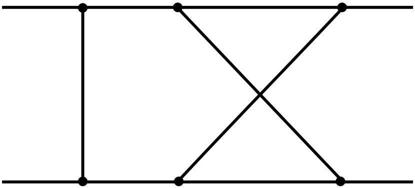
where Ω is an arbitrary subset of the lines $1,\ldots,L$, when the integration over the rest of the variables, i.e. for $i\notin\Omega$, is extended to the integration from zero to infinity.
One can prove this theorem in a simple way starting from the alpha representation using

$$1=\int_0^\infty\frac{d\lambda}{\lambda}\delta\left(1-\frac{1}{\lambda}\sum_{i=1}^{N}\alpha_i\right)\Leftrightarrow 1=\int_0^\infty\frac{d\lambda}{\lambda}\delta\left(1-\frac{1}{\lambda}\sum_{i\in\Omega}\alpha_i\right)$$

and change variables from α_i to $\alpha_i=\lambda x_i$ as shown above.

Non–Planar DoubleBox

J. B. Tausk,
"Nonplanar massless two loop Feynman diagrams with four on-shell legs",
Phys. Lett. B **469** (1999) 225; [hep-ph/9909506]



$$B_7^{NP} = \iint d^dk_1 d^dk_2 \frac{1}{[(k_1+k_2+p_1+p_2)^2]^{n_1}[(k_1+k_2+p_2)^2]^{n_2}[(k_1+K_2)^2]^{n_3}}$$
$$\frac{1}{[(k_1-p_3)^2]^{n_4}[(k_1)^2]^{n_5}[(k_2-p_4)^2]^{n_6}[(k_2)^2]^{n_7}}$$

$$U(x) = x[1]x[2] + x[1]x[4] + x[2]x[4] + x[1]x[5] + x[2]x[5] + x[2]x[6] + x[4]x[6] + x[5]x[6] \\ + x[1]x[7] + x[4]x[7] + x[5]x[7] + x[6]x[7]$$

$$F(x) = -s\ x[1]x[2]x[5] - s\ x[1]x[3]x[5] - s\ x[2]x[3]x[5] - u\ x[2]x[4]x[6] \\ - s\ x[3]x[5]x[6] - t\ x[1]x[4]x[7] - s\ x[3]x[5]x[7] - s\ x[3]x[6]x[7]$$

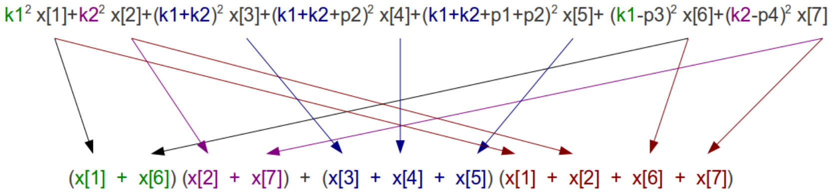


Figure: Factorization scheme

$$U(x) = (x[1]+x[6])(x[2]+x[7]) + (x[3]+x[4]+x[5])(x[1]+x[2]+x[6]+x[7])$$

$$F(x) = -t\,x[1]x[4]x[7] - u\,x[2]x[4]x[6] - s\,x[1]x[2]x[5] \\ - s\,x[3]x[6]x[7] - s\,x[3]x[5](x[1]+x[2]+x[6]+x[7])$$

Now we can apply the Cheng-Wu theorem and integrations will look as follows

$$B_7^{NP} = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - d\right)}{\Gamma(n_1)\ldots\Gamma(n_N)} \int\limits_0^\infty dx_3 dx_4 dx_5 \int\limits_0^1 dx_1 dx_2 dx_6 dx_7 \delta(1-(x_1+x_2+x_6+x_7)) \\ \frac{((x_1+x_6)(x_2+x_7)+x_3+x_4+x_5)^{N_\nu - \frac{3d}{2}}}{(-t\,x_1x_4x_7 - u\,x_2x_4x_6 - s\,x_1x_2x_5 - s\,x_3x_6x_7 - s\,x_3x_5)^{N_\nu - d}}$$

$$B_7^{NP} = \frac{(-1)^{N_\nu}}{\Gamma(n_1)\ldots\Gamma(n_N)} \int\limits_{-i\infty}^{i\infty} dz_1\ldots dz_4 \int dx_1\ldots dx_7\; (-s)^{-N_\nu+d-z_2-z_3}(-t)^{z_2}(-u)^{z_3} \\ \times \Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(N_\nu - d + z_1 + z_2 + z_3 + z_4) \\ \times x_1^{-N_\nu+d-z_1-z_2-z_3} x_2^{z_2+z_3} x_3{}^{-N_\nu+d-z_2-z_3-z_4} x_4{}^{z_1+z_3} x_5{}^{z_2+z_4} x_6^{z_1+z_2} x_7^{z_3+z_4} \\ \times (x_3+x_4+x_5+(x_1+x_6)(x_2+x_7))^{N_\nu - \frac{3d}{2}}$$

Integration over Cheng–Wu variables

$$\int_0^\infty dx\, x^{N_1}(x+A)^{N_2} = \frac{A^{1+N_1+N_2}\Gamma(1+N_1)\Gamma(-1-N_1-N_2)}{\Gamma(-N_2)}$$

4-dim result:

$$B_7^{NP} = \frac{(-1)^{N_\nu}}{\Gamma(n_1)\ldots\Gamma(n_7)} \int_{-i\infty}^{i\infty} dz_1 \ldots dz_4 (-s)^{4-2\epsilon-N_\nu-z_{23}}(-t)^{z_3}(-u)^{z_2}$$
$$\frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(2-\epsilon-n_{45})\Gamma(2-\epsilon-n_{67})}{\Gamma(4-2\epsilon-n_{4567})\Gamma(n_{45}+z_{1234})\Gamma(n_{67}+z_{1234})\Gamma(6-3\epsilon-N_\nu)}$$
$$\Gamma(n_2+z_{23})\Gamma(n_4+z_{24})\Gamma(n_5+z_{13})\Gamma(n_6+z_{34})\Gamma(n_7+z_{12})\Gamma^3(-2+\epsilon+n_{4567}+z_{1234})$$
$$\Gamma(4-2\epsilon-n_{124567}-z_{123})\Gamma(4-2\epsilon-n_{234567}-z_{234})\Gamma(-4+2\epsilon+N_\nu+z_{1234})$$

with notations $z_{i\ldots j\ldots k} = z_i + \ldots + z_j + \ldots + z_k$
and $n_{i\ldots j\ldots k} = n_i + \ldots + n_j + \ldots + n_k$

We make it another way, introducing suitable change of variables

$$B_7^{NP} = \iint d^dk_1 d^dk_2 \frac{1}{[(k_1+k_2+p_1+p_2+p_3)^2]^{n_1}[(k_1+k_2)^2]^{n_2}} \frac{1}{[(k_1)^2]^{n_3}[(k_1+p_1)^2]^{n_4}[(k_1+p_1+p_2)^2]^{n_5}[(k_2)^2]^{n_6}[(k_2+p_3)^2]^{n_7}}$$

$$\begin{aligned} m^2 = \sum x_i D_i &= x_1(k_1+k_2+p_1+p_2+p_3)^2 & x_1 &\to v_1 C_1 \\ &+ x_2(k_1+k_2)^2 & x_2 &\to v_1 C_2 \\ &+ x_3(k_1)^2 & x_3 &\to v_2 A_1 \\ &+ x_4(k_1+p_1)^2 & x_4 &\to v_2 A_2 \\ &+ x_5(k_1+p_1+p_2)^2 & x_5 &\to v_2 A_3 \\ &+ x_6(k_2)^2 & x_6 &\to v_3 B_1 \\ &+ x_7(k_2+p_3)^2 & x_7 &\to v_3 B_2 \end{aligned}$$

$$\delta\left(1-\sum_{i=1}^{7} x_i\right) \Rightarrow \delta(1-v_1-v_2-v_3)\delta(1-A_1-A_2-A_3)\delta(1-B_1-B_2)\delta(1-C_1-C_2)$$

Jacobian of the transformation:

$$J = v_1^{N_C-1} v_2^{N_A-1} v_3^{N_B-1} = v_1 v_2^2 v_3$$

1. Using $\delta(1-A_1-A_2-A_3)\delta(1-B_1-B_2)\delta(1-C_1-C_2)$ we can write for U and F

$$U = v_1v_2 + v_1v_3 + v_2v_3 \qquad F = -s\, A_1A_3v_1v_2^2 - s\, A_1A_3v_2^2v_3 - u\, A_2B_1C_1v_1v_2v_3$$
$$-s\, A_1B_2C_1v_1v_2v_3 - s\, A_3B_1C_2v_1v_2v_3 - t\, A_2B_2C_2v_1v_2v_3$$

2. Choose now v_2 as Cheng-Wu variable $\int_0^\infty dv_2 \int_0^1 dv_1dv_3\delta(1-v_1-v_3)$

$$U = v_2 + v_1v_3 \quad F = -s\, A_1A_3v_2^2 - u\, A_2B_1C_1v_1v_2v_3 - s\, A_1B_2C_1v_1v_2v_3$$
$$-s\, A_3B_1C_2v_1v_2v_3 - t\, A_2B_2C_2v_1v_2v_3$$

3. Apply MB relation for F
4. Integrate over v_2 using

$$\int_0^\infty dx\, x^{N_1}(x+A)^{N_2} = \frac{A^{1+N_1+N_2}\Gamma(1+N_1)\Gamma(-1-N_1-N_2)}{\Gamma(-N_2)}$$

5. Integrate over each subset of variables $\{v, A, B, C\}$ separately using

$$\int_0^1 \prod_{i=1}^N dx_i\, x_i^{n_i-1}\, \delta(1-x_1-\ldots-x_N) = \frac{\Gamma(n_1)\ldots\Gamma(n_N)}{\Gamma\left(n_1+\ldots+n_N\right)}$$

and get 4-dim representation, as before.

Some remarks

Change of variables in Symanzik polynomials U and F is effective as:

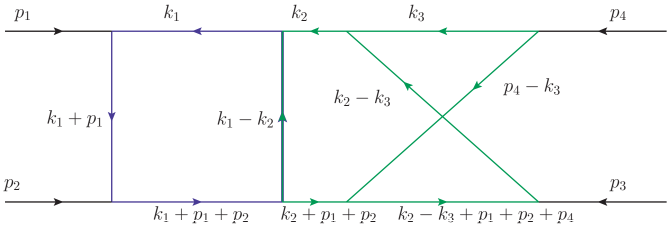
- They are homogeneous in the Feynman parameters, U is of degree L, F is of degree $L+1$
- U is linear in each Feynman parameter. If all internal masses are zero, then also F is linear in each Feynman parameter
- In expanded form each monomial of U has coefficient $+1$

Further:

1. Note that on a basic chart II, 3-loops cases, we have already $\mathrm{Length[U]==16}$ for GA (independently of both the multileg topology and mass configurations!), that is why much more effort must be done for simplifying U and F polynomials at this level
2. In general, it is not true that $dim(\mathtt{MB}[planars]) \simeq dim(\mathtt{MB}[non-planars])$,
3. Cases of massive external legs are completely different, e.g. such a factorization is not always the best:

$$F = F_0(\mathrm{scales!}) + \mathrm{U}\sum_{\mathrm{n=1}}^{\mathrm{N}} \mathrm{x_n}\frac{\mathrm{m_n^2}}{\{\mathrm{scales}\}^2}$$

Hybrid method for 3–loops



Hybrid method: LA– $\{k_1\}$; GA– $\{k_2, k_3\}$

Input:

```
PR[k1, 0, n1] PR[k1 + p1, 0, n2] PR[k1 + p1 + p2, 0, n3]
PR[k1 - k2, 0, n4] PR[k2, 0, n5] PR[k2 + p1 + p2, 0, n6]
PR[p1 + p2 + p4 + k2 - k3, 0,n7] PR[k2 - k3, 0, n8]
                        PR[k3, 0, n9] PR[p4 - k3, 0, n10]
```

step 1 input (LA– $\{k_1\}$):

```
PR[k1, 0, n1] PR[k1 + p1, 0, n2] PR[k1 + p1 + p2, 0, n3]
                                       PR[k1 - k2, 0, n4]
```

step 1 output:

```
((-1)^(2-eps-z3) (-s)^z3 Gamma[-z1] Gamma[2-eps-n1-n2-n4-z1-z2]
  Gamma[-z2] Gamma[n2+z2] Gamma[2-eps-n1-n2-n3-z3] Gamma[-z3]
     Gamma[n1+z1+z3] Gamma[-2+eps+n1+n2+n3+n4+z1+z2+z3])
/(Gamma[n1] Gamma[n2] Gamma[n3]
                         Gamma[4-2 eps-n1-n2-n3-n4] Gamma[n4])
PR[k2, 0, -z1] PR[k2+p1, 0, -z2]
                 PR[k2+p1+p2,0,-2+eps+n1+n2+n3+n4+z1+z2+z3]
```

step 2 input (GA– $\{k_2, k_3\}$):

```
   PR[k2, 0, n5-z1] PR[k2-k3, 0, n8] PR[k2+p1, 0, -z2]
     PR[k2+p1+p2, 0, -2+eps+n1+n2+n3+n4+n6+z1+z2+z3]
PR[k2-k3+p1+p2+p4, 0, n7] PR[k3, 0, n9] PR[p4-k3, 0, n10]
```

step 2 output ($n_i \to 1$):

```
((-s)^(-4 - 3 eps - z6 - z7) (-t)^z6 (-u)^
  z7 Gamma[-eps]^2 Gamma[-z1] Gamma[-1 - eps - z1 - z2]
  Gamma[1 + z2] Gamma[-1 - eps - z3] Gamma[-z3]
  Gamma[1 + z1 + z3]  Gamma[2 + eps + z1 + z2 + z3]
  Gamma[-2 - 2 eps - z3 - z4] Gamma[-z4] Gamma[-z5]
  Gamma[3 + eps + z1 + z2 + z3 + z4 + z5]
  Gamma[-3 - 3 eps - z3 - z4 - z5 - z6] Gamma[-z6]
  Gamma[1 + z5 + z6] Gamma[-3 - 3 eps - z3 - z4 - z5 - z7]
  Gamma[-3 - 3 eps - z1 - z3 - z5 - z6 - z7] Gamma[-z7]
  Gamma[1 + z5 + z7] Gamma[-z2 + z6 + z7]
  Gamma[4 + 3 eps + z3 + z4 + z5 + z6 + z7])/
(Gamma[-2 eps]^2 Gamma[1 - z1] Gamma[-2 - 4 eps - z3]
  Gamma[3 + eps + z1 + z2 + z3] Gamma[-2 - 3 eps - z3 - z4]^2)
```

Some non-planar MB box diagrams

Massless		Massive external legs	
2-loop	3-loop*	2-loop	3-loop*
6	9	>10	x
6	9	>8	x

AMBRE, LA: Dimensions of some $2 \rightarrow 2$ non–planar topologies before and after applying Barnes' first Lemma.

Massless		Massive external legs	
2-loop	3-loop*	2-loop	3-loop*
4	7	11	x
4	7	11	x

AMBRE, GA: Dimensions of some $2 \rightarrow 2$ non–planar topologies before and after applying Barnes' first Lemma.

* – diagram on slide 23

General structure of the MB integrals

$$\frac{1}{(2\pi i)^r}\int\limits_{-i\infty}^{i\infty}\dots\int\limits_{-i\infty}^{i\infty}\prod_i^r dz_i\ \mathbf{F}(Z,S,\vec{n},\epsilon)\frac{\prod_j \mathbf{G_j}(N_j)}{\prod_k \mathbf{G_k}(N_k)}$$

$$\begin{array}{rl}\mathbf{F}\text{ depends on:} & Z - \text{linear combinations of } r \text{ complex variables } z_i,\\ & S - \text{kinematic parameters and masses;}\\ & \vec{n}\in\{n_1,\dots,n_N\} - \text{powers of the } N \text{ propagators;}\\ & \\ \mathbf{G_i}: & \text{Gamma and PolyGamma functions}\\ N_i: & \text{linear combinations of } z_i,\ n_i \text{ and } \epsilon\end{array}$$

In practive F is a product of powers of S, with exponents being linear combinations of z_i, n_i:

$$\mathbf{F}\sim\prod_k X_k^{\prod_{i,j}(\alpha_i z_i+\beta_j n_j+\gamma\epsilon)}$$

$$\alpha_i,\beta_j,\gamma\in\mathbf{R},\quad e.g.\quad \mathbf{X}=\{\tfrac{s}{t},\tfrac{m^2}{s},\dots\}.$$

Changing MB integral into a sum: example

Two dimensional massive example:

```
int = MBint[((-s/m^2)^(z1 - z2) Gamma[1 - z1] Gamma[-z1]
      Gamma[z1] Gamma[1 + z1] Gamma[1 + z1 - z2]^2
      Gamma[-z2] Gamma[-z1 + z2])/Gamma[2 + z1 - 2 z2],
      {{eps -> 0}, {z1 -> -(13/32), z2 -> -(5/32)}}]
```

Input:

```
      MBIntToSum[int,kinematics,options]
e.g.  MBIntToSum[int,{s->-1, m->10},{z2 -> L, z1 -> L}]
```

Output:

```
{(B^n2 n2! (n1 + n2)! (HarmonicNumber[n1 + n2] -
     HarmonicNumber[2 + n1 + 2 n2]))/(2 + n1 + 2 n2)!,
     n1 >= 0 && n2 >= 0}
```

to be summed over `n1, n2`.
Another condition, e.g.: `n1 >= 0 && n2 >= 0 && n1 > n2.`

Changing MB integral into a sum: general remarks

`MBIntToSum[]`:

- Takes the `MBInt[]` (regulated, expanded in ϵ) and transforms it to the series, by closing the contours and calculating residues
- Depending on closing contours on left or right, suitable for further processing convergent infinite series can be obtained
- One of issues.

Simple one-dimensional example:

$$I = \int_{-i\infty-1/2}^{i\infty-1/2} dz_1 (-1/s)^{z_1} \frac{\Gamma[-z_1]^3 \Gamma[1+z_1]}{\Gamma[-2z_1]}$$

might look convergent for $s = -3$ when closing z_1 contour to the right and taking residues, but in fact the Gamma functions change the sum into a chain of alternating and increasing numbers (it converges above the threshold, $|s| > 4$)

Aim: Rewrite infinite sums in terms of iterated integrals

$$\sum_{n=0}^{\infty} f(n) x^n \quad \rightarrow \quad h_0(x) \int_0^x dt_1 h_1(t_1) \ldots \int_0^{t_{k-1}} dt_k h_k(t_k)$$

over suitably chosen alphabet of integrands

$$\frac{1}{t-a}, \quad \frac{1}{(t-a)\sqrt{t-b}}, \quad \frac{1}{\sqrt{t-b}\sqrt{t-c}}, \quad \frac{1}{(t-a)\sqrt{t-b}\sqrt{t-c}}$$

Main steps

1. Write summand in terms of nested sums using `Sigma` [29] by Schneider
2. Exploit summand structure by recursively applying rewrite rules to generate iterated integrals (Raab, cf. his talk at this conference)
3. Simplify the result to a canonical form (Raab, `HarmonicSums` [27] by Ablinger)

Rewrite rules

Well known basic rules to simplify the summands, e.g.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{f(n)}{n+1} x^n &= \frac{1}{x} \int_0^x dt \sum_{n=0}^{\infty} f(n) t^n \\
\sum_{n=1}^{\infty} \frac{f(n)}{n} x^n &= \int_0^x \frac{dt}{t} \sum_{n=1}^{\infty} f(n) t^n \\
\sum_{n=1}^{\infty} x^n \sum_{i=0}^{n} f(i) &= \frac{1}{1-x} \sum_{n=0}^{\infty} f(n) x^n
\end{aligned}$$

Derive new rules based on creative telescoping (Raab, using package `HolonomicFunctions` [28] by Koutschan), e.g.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{x^n}{(2n+1)\binom{2n}{n}} \sum_{i=0}^{n} f(i) &= \frac{2}{\sqrt{x}\sqrt{4-x}} \int_0^x \frac{dt}{\sqrt{t}\sqrt{4-t}} \sum_{n=0}^{\infty} t^n \frac{f(n)}{\binom{2n}{n}} \\
\sum_{n=0}^{\infty} x^n \binom{2n}{n} \sum_{i=0}^{n} f(i) &= \frac{1}{\sqrt{\frac{1}{4}-x}} \left(\frac{f(0)}{2} + \frac{1}{4} \int_0^x \frac{dt}{t\sqrt{\frac{1}{4}-t}} \sum_{n=0}^{\infty} t^n n \binom{2n}{n} f(n) \right)
\end{aligned}$$

Example

$$\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{\infty}\frac{n_2!(n_1+n_2)!}{(n_1+2n_2+2)!}\left(S_1(n_1+n_2)-S_1(n_1+2n_2+2)\right)x^{n_2}$$

1. Compute inner sum using `Sigma`

$$\sum_{n_2=0}^{\infty}\frac{S_1(n_2)-S_1(2n_2+1)}{(n_2+1)(2n_2+1)\binom{2n_2}{n_2}}x^{n_2}$$

2. Rewrite in terms of iterated integrals using our rewrite rules

$$\frac{1}{x}\int_0^x\frac{dt_1}{\sqrt{t_1}\sqrt{4-t_1}}\int_0^{t_1}\frac{dt_2}{\sqrt{t_2}\sqrt{4-t_2}}\left(-2+\int_0^{t_2}\frac{dt_3}{t_3^{3/2}\sqrt{4-t_3}}\int_0^{t_3}\frac{t_4dt_4}{\sqrt{t_4}\sqrt{4-t_4}}\right)$$

3. Bring to canonical form

$$-\frac{1}{x}\int_0^x\frac{dt_1}{\sqrt{t_1}\sqrt{4-t_1}}\int_0^{t_1}\frac{dt_2}{t_2}\int_0^{t_2}\frac{dt_3}{\sqrt{t_3}\sqrt{4-t_3}}$$

4. Remove square-roots by $x=-\frac{(1-y)^2}{y}$ and rewrite by `HarmonicSums`

$$\frac{y}{(1-y)^2}\left(4\zeta_3+2\zeta_2\ln(y)+\frac{\ln(y)^3}{6}+2\ln(y)\mathrm{Li}_2(y)-4\mathrm{Li}_3(y)\right)$$

Conclusions

1. MB integrals are exploited towards general analytic solutions

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2. Basic non-planar version of AMBRE is ready - Evgen Dubovyk

	massless diagrams:	100% d.o.s.;
$+$	massive propagators:	85 % d.o.s;
$+$	massive external legs:	25 % d.o.s.

Conclusions

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Conclusions

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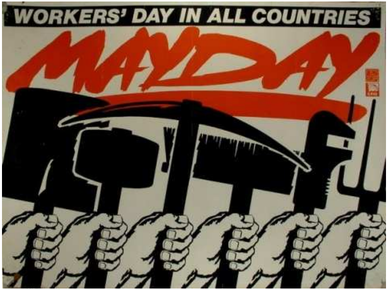
2. Basic non-planar version of AMBRE is ready - Evgen Dubovyk
   massless diagrams: 100% d.o.s.;
   $+$ massive propagators: 85 % d.o.s;
   $+$ massive external legs: 25 % d.o.s.
3. Changing MB integrals appropriate sums package is underway and tested - Michał Ochman
4. Work is undertaken on rewriting infinite sums for massive MB integrals into iterated integrals over suitably alphabet and by applying rewrite rules - DESY/Linz

Thank you for taking time!



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