# Examples, non-planar massless 2-loop vertex

## Case 1: Additional PR[k1 + p2, 0, n7], with n7=-1+eta

```
<< AMBREv3.1.1.m
```

```
AMBRE v3.1.1 [Apr 2017] by I.Dubovyk, http://us.edu.pl/~gluza/ambre/
License: http://creativecommons.org, CC BY-ND
Ref.: I. Dubovyk, J. Gluza, T. Riemann, J. Usovitsch, arXiv:1607.07538.
```

```
<< PlanarityTestv1.2.m
```

```
by E. Dubovyk and K. Bielas ver: 1.2
created: January 2014
last executed: 10.04.2017 at 17:49
```

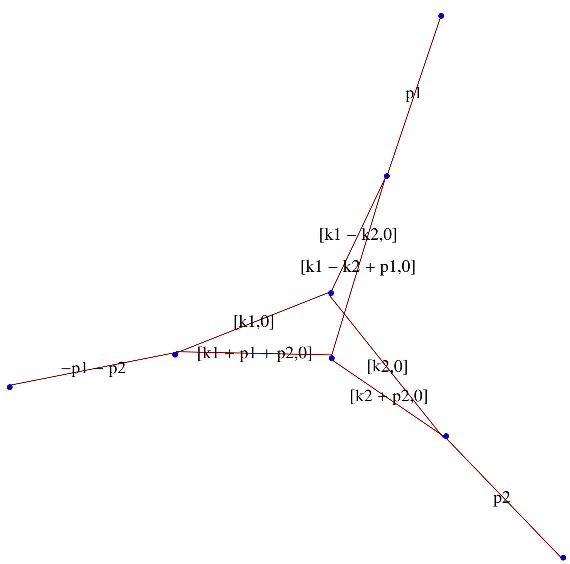
```
invariants = {p1^2 → 0, p2^2 → 0, p1 * p2 → s / 2};
```

```
prs = PR[k1, 0, n1] PR[k1 - k2, 0, n2] PR[k2, 0, n3]
   PR[k1 - k2 + p1, 0, n4] PR[k2 + p2, 0, n5] PR[k1 + p1 + p2, 0, n6];
```

```
PlanarityTest[{prs}, {k1, k2}, DrawGraph → True];
```

```
The Diagram

is non-planar.
```




```
res = MBreprNP[{1}, {prs * PR[k1 + p2, 0, n7]}, {k1, k2}]
```

Fauto::mode : F polynomial will be calculated in AUTO mode. In order to use MANUAL mode execute Fauto[0].

```
Upoly =
 x[1] x[2] + x[1] x[3] + x[2] x[3] + x[1] x[4] + x[3] x[4] + x[2] x[5] + x[3] x[5] + x[4] x[5] +
  x[1] x[6] + x[2] x[6] + x[4] x[6] + x[5] x[6] + x[2] x[7] + x[3] x[7] + x[4] x[7] + x[6] x[7]

Fpoly = s x[3] x[4] x[5] - s x[1] x[4] x[6] - s x[1] x[2] x[7] -
  s x[1] x[3] x[7] - s x[2] x[3] x[7] - s x[1] x[4] x[7] - s x[1] x[6] x[7]
```

$$\left\{\left((-1)^{n1+n2+n3+n4+n5+n6+n7}\,(-s)^{4-2\,eps-n1-n2-n3-n4-n5-n6-n7-z2}\,s^{z2}\,\mathrm{Gamma}[2-eps-n2-n4]\,\mathrm{Gamma}[2-eps-n3-n5]\,\mathrm{Gamma}[2-eps-n1-n6-n7-z1]\,\mathrm{Gamma}[-z1]\,\mathrm{Gamma}[-z2]\,\mathrm{Gamma}[n7+z2]\,\mathrm{Gamma}[4-2\,eps-n1-n2-n3-n5-n6-n7-z1-z3]\,\mathrm{Gamma}[4-2\,eps-n2-n3-n4-n5-n6-n7-z2-z3]\,\mathrm{Gamma}[4-2\,eps-n1-n2-n3-n4-n6-n7-z1-z2-z3]\,\mathrm{Gamma}[-z3]\,\mathrm{Gamma}[n2+z3]\,\mathrm{Gamma}[n6+z1+z3]\,\mathrm{Gamma}[n3+z2+z3]\,\mathrm{Gamma}[-4+2\,eps+n1+n2+n3+n4+n5+n6+n7+z1+z2+z3]\right)\Big/\left(\mathrm{Gamma}[n1]\,\mathrm{Gamma}[n2]\,\mathrm{Gamma}[n3]\,\mathrm{Gamma}[n4]\,\mathrm{Gamma}[4-2\,eps-n2-n3-n4-n5]\,\mathrm{Gamma}[n5]\,\mathrm{Gamma}[n6]\,\mathrm{Gamma}[6-3\,eps-n1-n2-n3-n4-n5-n6-n7]\,\mathrm{Gamma}[n7]\,\mathrm{Gamma}[4-2\,eps-n1-n2-n4-n6-n7-z1]\,\mathrm{Gamma}[4-2\,eps-n1-n3-n5-n6-n7-z1]\right)\right\}$$

**finres = res /. {n1 → 1, n2 → 1, n3 → 1, n4 → 1, n5 → 1, n6 → 1, n7 → -1 + eta}**

$$\left\{\left((-1)^{5+eta}\,(-s)^{-1-2\,eps-eta-z2}\,s^{z2}\,\mathrm{Gamma}[-eps]^2\,\mathrm{Gamma}[1-eps-eta-z1]\,\mathrm{Gamma}[-z1]\,\mathrm{Gamma}[-z2]\,\mathrm{Gamma}[-1+eta+z2]\,\mathrm{Gamma}[-2\,eps-eta-z1-z3]\,\mathrm{Gamma}[-2\,eps-eta-z2-z3]\,\mathrm{Gamma}[-2\,eps-eta-z1-z2-z3]\,\mathrm{Gamma}[-z3]\,\mathrm{Gamma}[1+z3]\,\mathrm{Gamma}[1+z1+z3]\,\mathrm{Gamma}[1+z2+z3]\,\mathrm{Gamma}[1+2\,eps+eta+z1+z2+z3]\right)\Big/\left(\mathrm{Gamma}[-2\,eps]\,\mathrm{Gamma}[1-3\,eps-eta]\,\mathrm{Gamma}[-1+eta]\,\mathrm{Gamma}[1-2\,eps-eta-z1]^2\right)\right\}$$

**(***
**In the above: $(-s)^{-1-2\ eps-eta-z2}\ s^{z2} = (-s)^{-1-2\ eps-eta} * (-1)^{-z2}$ ,**
**see after expansion - will remain just 1/s, as expected.**
***)**

```
<< MB.m
```

MB 1.2

by Michal Czakon

improvements by Alexander Smirnov

more info in hep-ph/0511200

last modified 2 Jan 09

**rules = MBoptimizedRules[finres[[1]], eta → 0, {}, {eps, eta}]**

MBrules::norules : no rules could be found to regulate this integral

MBrules::norules : no rules could be found to regulate this integral

MBrules::norules : no rules could be found to regulate this integral

General::stop : Further output of MBrules::norules will be suppressed during this calculation. ≫

$$\left\{\left\{eps \to -\frac{15}{64},\ eta \to \frac{37}{32}\right\},\ \left\{z1 \to -\frac{1}{4},\ z2 \to -\frac{1}{32},\ z3 \to -\frac{23}{32}\right\}\right\}$$

**Step1cont = MBcontinue[finres[[1]], eta → 0, rules];**

```
Level 1
Taking +residue in z2 = 1 - eta
Taking +residue in z2 = -eta
Taking +residue in z3 = -1 - 2 eps - eta - z1 - z2
Level 2
Integral {1}
Integral {2}
Integral {3}
Taking +residue in z2 = -eta
Level 3
Integral {3, 1}
5 integral(s) found
```

**after = MBexpand[Step1cont, 1, {eta, 0, 0}]**

$$\left\{\text{MBint}\left[-\left((-s)^{-2\,\text{eps}}\,\Gamma[-\text{eps}]^2\,\Gamma[1-\text{eps}-z1]\,\Gamma[-z1]\,\Gamma[-1-2\,\text{eps}-z3]\,\Gamma[-1-2\,\text{eps}-z1-z3]\,\Gamma[-2\,\text{eps}-z1-z3]\,\Gamma[-z3]\,\Gamma[1+z3]\,\Gamma[2+z3]\,\Gamma[1+z1+z3]\,\Gamma[2+2\,\text{eps}+z1+z3]\right)\Big/\left(s\,\Gamma[1-3\,\text{eps}]\,\Gamma[-2\,\text{eps}]\,\Gamma[1-2\,\text{eps}-z1]^2\right),\ \left\{\left\{\text{eps}\to-\frac{15}{64},\ \text{eta}\to 0\right\},\ \left\{z1\to-\frac{1}{4},\ z3\to-\frac{23}{32}\right\}\right\}\right],\right.$$

$$\text{MBint}\left[\left((-s)^{-2\,\text{eps}}\,\Gamma[-\text{eps}]^2\,\Gamma[1-\text{eps}-z1]\,\Gamma[-z1]\,\Gamma[-2\,\text{eps}-z3]\,\Gamma[-2\,\text{eps}-z1-z3]^2\,\Gamma[-z3]\,\Gamma[1+z3]^2\,\Gamma[1+z1+z3]\,\Gamma[1+2\,\text{eps}+z1+z3]\right)\Big/\left(s\,\Gamma[1-3\,\text{eps}]\,\Gamma[-2\,\text{eps}]\,\Gamma[1-2\,\text{eps}-z1]^2\right),\ \left\{\left\{\text{eps}\to-\frac{15}{64},\ \text{eta}\to 0\right\},\ \left\{z1\to-\frac{1}{4},\ z3\to-\frac{23}{32}\right\}\right\}\right],$$

$$\left.\text{MBint}\left[\left((-s)^{-2\,\text{eps}}\,\Gamma[-\text{eps}]^2\,\Gamma[-2\,\text{eps}-z1]^2\,\Gamma[1-\text{eps}-z1]\,\Gamma[-z1]\,\Gamma[1+z1]\,\Gamma[1+2\,\text{eps}+z1]\right)\Big/\left(s\,\Gamma[1-3\,\text{eps}]\,\Gamma[1-2\,\text{eps}-z1]^2\right),\ \left\{\left\{\text{eps}\to-\frac{15}{64},\ \text{eta}\to 0\right\},\ \left\{z1\to-\frac{1}{4}\right\}\right\}\right]\right\}$$

**integrals = after /. MBint[integrand_, rules_] :→ MBcontinue[integrand, eps → 0, rules]**

```
Level 1
Taking -residue in z3 = -1 - 2 eps
Taking -residue in z3 = -1 - 2 eps - z1
Level 2
Integral {1}
Taking +residue in z1 = 2 eps
Integral {2}
Level 3
Integral {1, 1}
4 integral(s) found
Level 1
Taking -residue in z3 = -1 - 2 eps - z1
Level 2
Integral {1}
2 integral(s) found
Level 1
1 integral(s) found
```

```
{{{{MBint[-((-s)^(-2 eps) Gamma[1 - 2 eps]^2 Gamma[-2 eps]^2 Gamma[-eps]^2 Gamma[1 + 2 eps]^2) /
      (s Gamma[1 - 4 eps]^2), {{eps → 0, eta → 0}, {}}]},
   MBint[-((-s)^(-2 eps) Gamma[1 - 2 eps] Gamma[-eps]^2 Gamma[1 + 2 eps] Gamma[1 - z1]
        Gamma[1 - eps - z1] Gamma[-z1]^2 Gamma[1 + z1] Gamma[-2 eps + z1]) /
      (s Gamma[1 - 3 eps] Gamma[1 - 2 eps - z1]^2), {{eps → 0, eta → 0}, {z1 → -1/4}}]},
  {MBint[-((-s)^(-2 eps) Gamma[-eps]^2 Gamma[-2 eps - z1] Gamma[1 - eps - z1]
        Gamma[-z1] Gamma[z1] Gamma[1 + 2 eps + z1]) /
      (s Gamma[1 - 3 eps] Gamma[1 - 2 eps - z1]), {{eps → 0, eta → 0}, {z1 → -1/4}}]},
  MBint[-((-s)^(-2 eps) Gamma[-eps]^2 Gamma[1 - eps - z1] Gamma[-z1] Gamma[-1 - 2 eps - z3]
       Gamma[-1 - 2 eps - z1 - z3] Gamma[-2 eps - z1 - z3] Gamma[-z3]
       Gamma[1 + z3] Gamma[2 + z3] Gamma[1 + z1 + z3] Gamma[2 + 2 eps + z1 + z3]) /
     (s Gamma[1 - 3 eps] Gamma[-2 eps] Gamma[1 - 2 eps - z1]^2),
    {{eps → 0, eta → 0}, {z1 → -1/4, z3 → -23/32}}]},
 {{MBint[-((-s)^(-2 eps) Gamma[-eps]^2 Gamma[-2 eps - z1]^2 Gamma[1 - eps - z1]
        Gamma[-z1] Gamma[1 + z1] Gamma[1 + 2 eps + z1]) /
      (s Gamma[1 - 3 eps] Gamma[1 - 2 eps - z1]^2), {{eps → 0, eta → 0}, {z1 → -1/4}}]},
  MBint[((-s)^(-2 eps) Gamma[-eps]^2 Gamma[1 - eps - z1] Gamma[-z1]
       Gamma[-2 eps - z3] Gamma[-2 eps - z1 - z3]^2 Gamma[-z3]
       Gamma[1 + z3]^2 Gamma[1 + z1 + z3] Gamma[1 + 2 eps + z1 + z3]) /
     (s Gamma[1 - 3 eps] Gamma[-2 eps] Gamma[1 - 2 eps - z1]^2),
    {{eps → 0, eta → 0}, {z1 → -1/4, z3 → -23/32}}]},
 {MBint[((-s)^(-2 eps) Gamma[-eps]^2 Gamma[-2 eps - z1]^2 Gamma[1 - eps - z1]
       Gamma[-z1] Gamma[1 + z1] Gamma[1 + 2 eps + z1]) /
     (s Gamma[1 - 3 eps] Gamma[1 - 2 eps - z1]^2), {{eps → 0, eta → 0}, {z1 → -1/4}}]}}
```

```
ser = MBexpand[integrals, Exp[2 * eps * EulerGamma], {eps, 0, 0}];
```

```
MBanalytic = MBmerge[ser]
```

```
{MBint[1/(480 eps^4 s) (-120 + 60 eps^2 π^2 + 123 eps^4 π^4 + 120 eps^2 (-2 + eps^2 π^2) Log[-s]^2 +
      160 eps^3 Log[-s]^3 - 80 eps^4 Log[-s]^4 - 2200 eps^3 PolyGamma[2, 1] +
      40 eps Log[-s] (6 - 3 eps^2 π^2 + 110 eps^3 PolyGamma[2, 1])),
  {{eps → 0, eta → 0}, {}}], MBint[
  -1/(2 eps^2 s) Gamma[-z1]^2 Gamma[z1] Gamma[1 + z1] (4 + 4 eps EulerGamma + 2 eps^2 EulerGamma^2 -
       eps^2 π^2 - 8 eps Log[-s] - 8 eps^2 EulerGamma Log[-s] + 8 eps^2 Log[-s]^2 +
       10 eps^2 PolyGamma[0, 1 - z1]^2 + 4 eps^2 PolyGamma[0, -z1]^2 - 4 eps PolyGamma[0, z1] -
       4 eps^2 EulerGamma PolyGamma[0, z1] + 8 eps^2 Log[-s] PolyGamma[0, z1] +
       4 eps^2 PolyGamma[0, z1]^2 + 4 eps PolyGamma[0, 1 + z1] +
       4 eps^2 EulerGamma PolyGamma[0, 1 + z1] - 8 eps^2 Log[-s] PolyGamma[0, 1 + z1] +
       4 eps^2 PolyGamma[0, 1 + z1]^2 + 4 eps PolyGamma[0, 1 - z1]
         (2 + 2 eps EulerGamma - 4 eps Log[-s] - eps PolyGamma[0, -z1] -
            3 eps PolyGamma[0, z1] + eps PolyGamma[0, 1 + z1]) - 4 eps PolyGamma[0, -z1]
         (1 + eps EulerGamma - 2 eps Log[-s] + 2 eps PolyGamma[0, 1 + z1]) -
       10 eps^2 PolyGamma[1, 1 - z1] + 4 eps^2 PolyGamma[1, -z1] +
       4 eps^2 PolyGamma[1, z1] + 4 eps^2 PolyGamma[1, 1 + z1]),
  {{eps → 0, eta → 0}, {z1 → -1/4}}], MBint[
  1/(eps s Gamma[1 - z1])
   2 Gamma[-z1] Gamma[-z1 - z3]
    Gamma[-z3] Gamma[1 + z3] Gamma[1 + z1 + z3]
    (Gamma[-z1 - z3] Gamma[-z3] Gamma[1 + z3] Gamma[1 + z1 + z3]
        (-1 + eps EulerGamma + 2 eps Log[-s] - 3 eps PolyGamma[0, 1 - z1] + 4 eps PolyGamma[
            0, -z1 - z3] + 2 eps PolyGamma[0, -z3] - 2 eps PolyGamma[0, 1 + z1 + z3]) -
      Gamma[-1 - z3] Gamma[-1 - z1 - z3] Gamma[2 + z3] Gamma[2 + z1 + z3]
        (-1 + eps EulerGamma + 2 eps Log[-s] - 3 eps PolyGamma[0, 1 - z1] +
          2 eps PolyGamma[0, -1 - z3] + 2 eps PolyGamma[0, -1 - z1 - z3] +
          2 eps PolyGamma[0, -z1 - z3] - 2 eps PolyGamma[0, 2 + z1 + z3])),
  {{eps → 0, eta → 0}, {z1 → -1/4, z3 → -23/32}}]}
```

```
MBintegrate[MBanalytic, {s → -1}]
```

```
Shifting contours...
Performing 3 lower-dimensional integrations with NIntegrate...1...2...3
Higher-dimensional integrals
Preparing MBpart1eps0 (dim 2)
Preparing MBpart2eps-1 (dim 2)
Running MBpart1eps0
Running MBpart2eps-1
```

$$\left\{-20.0907+\frac{0.25}{\text{eps}^4}-\frac{4.52357}{\text{eps}^2}-\frac{13.423}{\text{eps}},\left\{0.00178493+\frac{6.60224\times10^{-14}}{\text{eps}},0\right\}\right\}$$

```
Quit[]
```

```
FIESTAPath = "your fiesta path"; Get[FIESTAPath <> "FIESTA4.m"];
UsingQLink = True; UsingC = True;
CurrentIntegratorSettings = {{"maxeval", "500000"}};
```

```
SDEvaluate[UF[{k1, k2}, {-k1^2, -(k1 - k2)^2, -k2^2, -(k1 - k2 + p1)^2, -(k2 + p2)^2,
   -(k1 + p1 + p2)^2, -(k1 + p2)^2}, {p1^2 → 0, p2^2 → 0, p1 * p2 → s / 2,
   s → -1}], {1, 1, 1, 1, 1, 1, -1}, 0]
```

```
FIESTA 4.1
Current integrator: vegasCuba
CurrentIntegratorSettings: {{"epsrel","1.000000E-05"},{"epsabs","1.000000E-12"},{"mineval"
Integration test passed
Starting 1 subkernels
Subkernel will be used for launching external programs, all evaluations go on main kernel.
UsingC: True
NumberOfLinks: 1
UsingQLink: True
Strategy: STRATEGY_S
Integration has to be performed up to order 0
KLink created (2013 version)! You can read information on QOpen, QRead, QRemoveDatabase, Q(
Sector decomposition - 6 sectors
Primary sector 1 resulted in 14 sectors.
Primary sector 2 resulted in 14 sectors.
Primary sector 3 resulted in 14 sectors.
Primary sector 4 resulted in 14 sectors.
Primary sector 5 resulted in 14 sectors.
Primary sector 6 resulted in 14 sectors.
Totally: 0.3311 seconds; 12 sectors.
Preparing database: 0.0282 seconds.
Variable substitution..........0.4831 seconds; 168 terms.
Pole resolution..........0.5985 seconds; 444 terms.
Expression preparation..........0.495 seconds; 444 terms.
Epsilon expansion..........0.8898 seconds; 860 terms.
Preparing integration strings..........2.3979 seconds; 860 terms.
Database ready for integration.
Terms of order -4: 4, max vars: 1
Integrating..........0.0013 seconds.
Returned answer: -0.25 + pm* 1.*^-6
(-0.25 + 1.*^-6*pm1)*ep^(-4)
Terms of order -3: 18, max vars: 2
Integrating..........0.0015 seconds.
Returned answer: -4.*^-6 + pm* 0.00004
(-0.25 + 1.*^-6*pm2)*ep^(-4)+(0)*ep^(-3)
Terms of order -2: 46, max vars: 3
Integrating..........0.0009 seconds.
Returned answer: 5.346019 + pm* 0.000375
(-0.25 + 1.*^-6*pm4)*ep^(-4)+(0)*ep^(-3)+(4.523552 + 0.000375*pm6)*ep^(-2)
Terms of order -1: 126, max vars: 4
Integrating..........0.001 seconds.
Returned answer: 12.621554 + pm* 0.004803
(-0.25 + 1.*^-6*pm7)*ep^(-4)+(0)*ep^(-3)+(4.523552 + 0.000375*pm9)*ep^(-2)+(13.422913 + 0.(
Terms of order 0: 168, max vars: 5
Integrating..........0.0009 seconds.
Returned answer: 4.933841 + pm* 0.018563
(-0.25 + 1.*^-6*pm11)*ep^(-4)+(0)*ep^(-3)+(4.523552 + 0.000375*pm13)*ep^(-2)+(13.422913 + (
Total integration time: 81.5379
Total time used: 87.0677 seconds.
```

$$20.0863 + \frac{-0.25 + 1. \times 10^{-6}\,\mathrm{pm16}}{\mathrm{ep}^4} + \frac{4.52355 + 0.000375\,\mathrm{pm18}}{\mathrm{ep}^2} + \frac{13.4229 + 0.004804\,\mathrm{pm19}}{\mathrm{ep}} + 0.018604\,\mathrm{pm20}$$

```
(*
Overall sign difference with MB due
  to different convention in propagators definition.
   Also be careful with using Fiesta and MB-suite in one Mathematica session,
may lead to errors
*)
```

```
Quit[]
```

## Case 2: PR[k2 + p1, 0, n7], with n7=-1+eta

```
<< AMBREv3.1.1.m
```

```
AMBRE v3.1.1 [Apr 2017] by I.Dubovyk, http://us.edu.pl/~gluza/ambre/
License: http://creativecommons.org, CC BY-ND
Ref.: I. Dubovyk, J. Gluza, T. Riemann, J. Usovitsch, arXiv:1607.07538.
```

```
<< PlanarityTestv1.2.m
```

```
by E. Dubovyk and K. Bielas ver: 1.2
created: January 2014
last executed: 10.04.2017 at 17:49
```

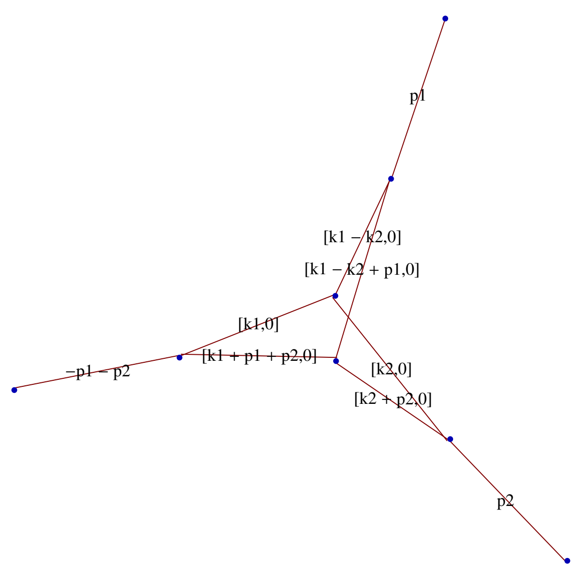
```
invariants = {p1^2 → 0, p2^2 → 0, p1 * p2 → s / 2};
```

```
prs = PR[k1, 0, n1] PR[k1 - k2, 0, n2] PR[k2, 0, n3]
   PR[k1 - k2 + p1, 0, n4] PR[k2 + p2, 0, n5] PR[k1 + p1 + p2, 0, n6];
```

```
PlanarityTest[{prs}, {k1, k2}, DrawGraph → True];
```

```
The Diagram

is non-planar.
```



```
res = MBreprNP[{1}, {prs * PR[k2 + p1, 0, n7]}, {k1, k2}]
```

Fauto::mode : F polynomial will be calculated in AUTO mode. In order to use MANUAL mode execute Fauto[0].

```
Upoly =
 x[1] x[2] + x[1] x[3] + x[2] x[3] + x[1] x[4] + x[3] x[4] + x[1] x[5] + x[2] x[5] + x[4] x[5] +
  x[1] x[6] + x[2] x[6] + x[4] x[6] + x[2] x[7] + x[3] x[7] + x[4] x[7] + x[5] x[7] + x[6] x[7]

Fpoly = - s x[1] x[4] x[6] + s x[1] x[5] x[6] + s x[2] x[5] x[6] + s x[4] x[5] x[6] -
  s x[1] x[2] x[7] - s x[1] x[3] x[7] - s x[2] x[3] x[7] - s x[1] x[4] x[7] -
  s x[1] x[5] x[7] + s x[4] x[5] x[7] - s x[1] x[6] x[7] + s x[5] x[6] x[7]
```

$\{((-1)^{n1+n2+n3+n4+n5+n6+n7}\ (-s)^{z1+z2+z3+z5}\ s^{4-2\,eps-n1-n2-n3-n4-n5-n6-n7-z1-z2-z3-z5}$

```
    Gamma[-z1] Gamma[2 - eps - n1 - n6 - z1 - z2] Gamma[-z2]
    Gamma[2 - eps - n2 - n4 + z2] Gamma[-z3] Gamma[n2 + z3] Gamma[n3 + z3]
    Gamma[4 - 2 eps - n1 - n2 - n3 - n4 - n6 - n7 - z1 - z2 - z3 - z4] Gamma[-z4]
    Gamma[n6 + z1 + z2 + z3 + z4] Gamma[4 - 2 eps - n1 - n2 - n3 - n4 - n5 - n6 - z1 - z2 - z3 - z5]
    Gamma[8 - 4 eps - 2 n1 - 2 n2 - n3 - 2 n4 - n5 - 2 n6 - n7 - 2 z1 - z2 - z3 - z4 - z5]
    Gamma[-z5] Gamma[n1 + z1 + z2 + z5] Gamma[n4 + z4 + z5]
    Gamma[-4 + 2 eps + n1 + n2 + n3 + n4 + n5 + n6 + n7 + z1 + z2 + z3 + z4 + z5]
    Gamma[-2 + eps + n1 + n2 + n4 + n6 + 2 z1 + z2 + z3 + z4 + z5]) /
  (Gamma[n1] Gamma[n2] Gamma[n3] Gamma[n4] Gamma[n5] Gamma[n6]
    Gamma[6 - 3 eps - n1 - n2 - n3 - n4 - n5 - n6 - n7]
    Gamma[n7] Gamma[4 - 2 eps - n1 - n2 - n4 - n6 - z1]
    Gamma[8 - 4 eps - 2 n1 - 2 n2 - n3 - 2 n4 - n5 - 2 n6 - n7 - 2 z1 - 2 z2 - z3 - z4 - z5]
    Gamma[n2 + n4 + z3 + z4 + z5] Gamma[n1 + n6 + 2 z1 + 2 z2 + z3 + z4 + z5])}
```

```
finres = res /. {n1 → 1, n2 → 1, n3 → 1, n4 → 1, n5 → 1, n6 → 1, n7 → -1 + eta}
```

$\{((-1)^{5+eta}\ (-s)^{z1+z2+z3+z5}\ s^{-1-2\,eps-eta-z1-z2-z3-z5}$ Gamma[-z1]

```
    Gamma[-eps - z1 - z2] Gamma[-z2] Gamma[-eps + z2] Gamma[-z3] Gamma[1 + z3]^2
    Gamma[-2 eps - eta - z1 - z2 - z3 - z4] Gamma[-z4] Gamma[1 + z1 + z2 + z3 + z4]
    Gamma[-2 - 2 eps - z1 - z2 - z3 - z5] Gamma[-1 - 4 eps - eta - 2 z1 - z2 - z3 - z4 - z5]
    Gamma[-z5] Gamma[1 + z1 + z2 + z5] Gamma[1 + z4 + z5]
    Gamma[1 + 2 eps + eta + z1 + z2 + z3 + z4 + z5] Gamma[2 + eps + 2 z1 + z2 + z3 + z4 + z5]) /
  (Gamma[1 - 3 eps - eta] Gamma[-1 + eta] Gamma[-2 eps - z1]
    Gamma[-1 - 4 eps - eta - 2 z1 - 2 z2 - z3 - z4 - z5]
    Gamma[2 + z3 + z4 + z5] Gamma[2 + 2 z1 + 2 z2 + z3 + z4 + z5])}
```

```
<< MB.m
```

MB 1.2

by Michal Czakon

improvements by Alexander Smirnov

more info in hep-ph/0511200

last modified 2 Jan 09

```
rules = MBoptimizedRules[finres[[1]], eta → 0, {z2 > z1}, {eps, eta}]
```

MBrules::norules : no rules could be found to regulate this integral

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General::stop : Further output of MBrules::norules will be suppressed during this calculation. ≫

$$\left\{\left\{eps \to -\frac{67}{192},\ eta \to \frac{175}{128}\right\},\ \left\{z1 \to -\frac{1}{6},\ z2 \to -\frac{1}{12},\ z3 \to -\frac{101}{192},\ z4 \to -\frac{5}{48},\ z5 \to -\frac{37}{64}\right\}\right\}$$

```
Step1cont = MBcontinue[finres[[1]], eta → 0, rules];
```

```
Level 1
Taking +residue in z5 = -1 - 2 eps - eta - z1 - z2 - z3 - z4
Taking +residue in z5 = -2 - 2 eps - eta - z1 - z2 - z3 - z4
Level 2
Integral {1}
Taking +residue in z4 = 1 - eta
Taking +residue in z4 = -eta
Taking +residue in z4 = -1 - 2 eps - eta - z1 - z2 - z3
Integral {2}
Taking +residue in z4 = -eta
Level 3
Integral {1, 1}
Integral {1, 2}
Integral {1, 3}
Taking +residue in z3 = -1 - 2 eps - eta - z1 - z2
Integral {2, 1}
Level 4
Integral {1, 3, 1}
8 integral(s) found
```

```
after = MBexpand[Step1cont, 1, {eta, 0, 0}]
```

```
{MBint[-((-s)^(-2 eps) Gamma[-z1] Gamma[1 - eps + z1]
      Gamma[-eps - z1 - z2] Gamma[-z2] Gamma[-eps + z2] Gamma[-1 - 2 eps - z3]
      Gamma[-1 - 2 eps - z1 - z2 - z3] Gamma[-2 eps - z1 - z2 - z3] Gamma[-z3] Gamma[1 + z3]^2
      Gamma[2 + z1 + z2 + z3] Gamma[2 + 2 eps + z1 + z2 + z3]) / (s Gamma[1 - 3 eps]
      Gamma[-2 eps - z1 - z2] Gamma[1 - 2 eps - z1 - z2] Gamma[1 - 2 eps + z1 + z2]),
  {{eps -> -67/192, eta -> 0}, {z1 -> -1/6, z2 -> -1/12, z3 -> -101/192}}],
 MBint[((-s)^(-2 eps) Gamma[-z1] Gamma[1 - eps + z1] Gamma[-eps - z1 - z2] Gamma[-z2]
     Gamma[-eps + z2] Gamma[-2 eps - z3] Gamma[-2 eps - z1 - z2 - z3]^2 Gamma[-z3]
     Gamma[1 + z3]^2 Gamma[1 + z1 + z2 + z3] Gamma[1 + 2 eps + z1 + z2 + z3]) /
   (s Gamma[1 - 3 eps] Gamma[-2 eps - z1 - z2] Gamma[1 - 2 eps - z1 - z2]
     Gamma[1 - 2 eps + z1 + z2]),
  {{eps -> -67/192, eta -> 0}, {z1 -> -1/6, z2 -> -1/12, z3 -> -101/192}}],
 MBint[((-s)^(-2 eps) Gamma[-2 eps] Gamma[-z1] Gamma[1 - eps + z1]
     Gamma[-2 eps - z1 - z2] Gamma[-eps - z1 - z2] Gamma[-z2]
     Gamma[-eps + z2] Gamma[1 + z1 + z2] Gamma[1 + 2 eps + z1 + z2]) /
   (s Gamma[1 - 3 eps] Gamma[1 - 2 eps - z1 - z2] Gamma[1 - 2 eps + z1 + z2]),
  {{eps -> -67/192, eta -> 0}, {z1 -> -1/6, z2 -> -1/12}}],
 MBint[-((-s)^(-2 eps) Gamma[1 - 2 eps - z1] Gamma[-z1] Gamma[-eps + z1] Gamma[-eps - z1 - z2]
      Gamma[-z2] Gamma[-eps + z2] Gamma[-1 - 2 eps - z3] Gamma[-1 - 2 eps - z1 - z2 - z3]
      Gamma[-2 eps - z1 - z2 - z3] Gamma[-z3] Gamma[1 + z3]^2 Gamma[1 + z1 + z2 + z3]
      Gamma[2 + 2 eps + z1 + z2 + z3]) / (s Gamma[1 - 3 eps] Gamma[-2 eps - z1]
      Gamma[-2 eps - z1 - z2] Gamma[1 - 2 eps - z1 - z2] Gamma[-2 eps + z1 + z2]),
  {{eps -> -67/192, eta -> 0}, {z1 -> -1/6, z2 -> -1/12, z3 -> -101/192}}]}
```

```
integrals = after /. MBint[integrand_, rules_] :> MBcontinue[integrand, eps -> 0, rules]
```

```
Level 1
Taking +residue in z2 = eps
Taking -residue in z3 = -1 - 2 eps
Taking -residue in z3 = -1 - 2 eps - z1 - z2
Level 2
Integral {1}
Integral {2}
Taking +residue in z2 = eps
Integral {3}
Taking +residue in z2 = eps
Level 3
```

```
Integral {2, 1}

Integral {3, 1}

6 integral(s) found

Level 1

Taking +residue in z2 = eps

Taking -residue in z3 = -1 - 2 eps - z1 - z2

Level 2

Integral {1}

Integral {2}

Taking +residue in z2 = eps

Level 3

Integral {2, 1}

4 integral(s) found

Level 1

Taking +residue in z2 = eps

Level 2

Integral {1}

2 integral(s) found

Level 1

Taking +residue in z1 = eps

Taking +residue in z2 = eps

Taking -residue in z3 = -1 - 2 eps

Taking -residue in z3 = -1 - 2 eps - z1 - z2

Level 2

Integral {1}

Taking -residue in z3 = -1 - 3 eps - z2

Integral {2}

Integral {3}

Taking +residue in z1 = eps

Taking +residue in z2 = eps

Integral {4}

Taking +residue in z2 = eps

Level 3

Integral {1, 1}

Integral {3, 1}

Taking +residue in z2 = eps
```

```
Integral {3, 2}

Integral {4, 1}

Level 4

Integral {3, 1, 1}

10 integral(s) found

{{{MBint[-((-s)^(-2 eps) Gamma[-eps] Gamma[-2 eps - z1] Gamma[-z1]
        Gamma[-1 - 2 eps - z3] Gamma[-1 - 3 eps - z1 - z3] Gamma[-3 eps - z1 - z3]
        Gamma[-z3] Gamma[1 + z3]^2 Gamma[2 + eps + z1 + z3] Gamma[2 + 3 eps + z1 + z3]) /
      (s Gamma[1 - 3 eps] Gamma[-3 eps - z1] Gamma[1 - 3 eps - z1]),
    {{eps -> 0, eta -> 0}, {z1 -> -1/6, z3 -> -101/192}}]}},
  {{MBint[-((-s)^(-2 eps) Gamma[-2 eps]^2 Gamma[-eps] Gamma[1 + 2 eps] Gamma[-2 eps - z1]
        Gamma[-eps - z1] Gamma[1 - eps - z1] Gamma[-z1] Gamma[1 - eps + z1] Gamma[
         1 + eps + z1]) / (s Gamma[1 - 3 eps] Gamma[-3 eps - z1] Gamma[1 - 3 eps - z1]),
     {{eps -> 0, eta -> 0}, {z1 -> -1/6}}]}, MBint[-((-s)^(-2 eps) Gamma[-2 eps]^2
        Gamma[1 + 2 eps] Gamma[-z1] Gamma[1 - eps + z1] Gamma[-z1 - z2] Gamma[1 - z1 - z2]
        Gamma[-eps - z1 - z2] Gamma[-z2] Gamma[-eps + z2] Gamma[1 + z1 + z2]) /
      (s Gamma[1 - 3 eps] Gamma[-2 eps - z1 - z2] Gamma[1 - 2 eps - z1 - z2]),
    {{eps -> 0, eta -> 0}, {z1 -> -1/6, z2 -> -1/12}}]},
  {{MBint[-((-s)^(-2 eps) Gamma[1 - 2 eps] Gamma[-eps] Gamma[-3 eps - z1]
        Gamma[-2 eps - z1] Gamma[-z1] Gamma[eps + z1] Gamma[1 + 3 eps + z1]) /
       (s Gamma[1 - 3 eps] Gamma[1 - 3 eps - z1]), {{eps -> 0, eta -> 0}, {z1 -> -1/6}}]},
   MBint[-((-s)^(-2 eps) Gamma[1 - 2 eps] Gamma[-z1] Gamma[1 - eps + z1]
        Gamma[-2 eps - z1 - z2] Gamma[-eps - z1 - z2] Gamma[-z2]
        Gamma[-eps + z2] Gamma[z1 + z2] Gamma[1 + 2 eps + z1 + z2]) /
      (s Gamma[1 - 3 eps] Gamma[1 - 2 eps - z1 - z2] Gamma[1 - 2 eps + z1 + z2]),
    {{eps -> 0, eta -> 0}, {z1 -> -1/6, z2 -> -1/12}}]},
  MBint[-((-s)^(-2 eps) Gamma[-z1] Gamma[1 - eps + z1] Gamma[-eps - z1 - z2] Gamma[-z2]
       Gamma[-eps + z2] Gamma[-1 - 2 eps - z3] Gamma[-1 - 2 eps - z1 - z2 - z3]
       Gamma[-2 eps - z1 - z2 - z3] Gamma[-z3] Gamma[1 + z3]^2
       Gamma[2 + z1 + z2 + z3] Gamma[2 + 2 eps + z1 + z2 + z3]) / (s Gamma[1 - 3 eps]
       Gamma[-2 eps - z1 - z2] Gamma[1 - 2 eps - z1 - z2] Gamma[1 - 2 eps + z1 + z2]),
   {{eps -> 0, eta -> 0}, {z1 -> -1/6, z2 -> -1/12, z3 -> -101/192}}]},
 {{MBint[((-s)^(-2 eps) Gamma[-eps] Gamma[-2 eps - z1] Gamma[-z1]
       Gamma[-2 eps - z3] Gamma[-3 eps - z1 - z3]^2 Gamma[-z3]
       Gamma[1 + z3]^2 Gamma[1 + eps + z1 + z3] Gamma[1 + 3 eps + z1 + z3]) /
     (s Gamma[1 - 3 eps] Gamma[-3 eps - z1] Gamma[1 - 3 eps - z1]),
```

```
    {{eps → 0, eta → 0}, {z1 → -1/6, z3 → -101/192}}]]},
  {{MBint[-((-s)^(-2 eps) Gamma[-2 eps] Gamma[-eps] Gamma[-3 eps - z1]
          Gamma[-2 eps - z1] Gamma[-z1] Gamma[1 + eps + z1] Gamma[1 + 3 eps + z1]) /
        (s Gamma[1 - 3 eps] Gamma[1 - 3 eps - z1]), {{eps → 0, eta → 0}, {z1 → -1/6}}]},
   MBint[-((-s)^(-2 eps) Gamma[-2 eps] Gamma[-z1] Gamma[1 - eps + z1]
         Gamma[-2 eps - z1 - z2] Gamma[-eps - z1 - z2] Gamma[-z2]
         Gamma[-eps + z2] Gamma[1 + z1 + z2] Gamma[1 + 2 eps + z1 + z2]) /
       (s Gamma[1 - 3 eps] Gamma[1 - 2 eps - z1 - z2] Gamma[1 - 2 eps + z1 + z2]),
    {{eps → 0, eta → 0}, {z1 → -1/6, z2 → -1/12}}]],
  MBint[((-s)^(-2 eps) Gamma[-z1] Gamma[1 - eps + z1] Gamma[-eps - z1 - z2] Gamma[-z2]
       Gamma[-eps + z2] Gamma[-2 eps - z3] Gamma[-2 eps - z1 - z2 - z3]^2 Gamma[-z3]
       Gamma[1 + z3]^2 Gamma[1 + z1 + z2 + z3] Gamma[1 + 2 eps + z1 + z2 + z3]) /
     (s Gamma[1 - 3 eps] Gamma[-2 eps - z1 - z2] Gamma[1 - 2 eps - z1 - z2]
       Gamma[1 - 2 eps + z1 + z2]),
   {{eps → 0, eta → 0}, {z1 → -1/6, z2 → -1/12, z3 → -101/192}}]]},
 {{MBint[((-s)^(-2 eps) Gamma[-2 eps] Gamma[-eps] Gamma[-3 eps - z1]
        Gamma[-2 eps - z1] Gamma[-z1] Gamma[1 + eps + z1] Gamma[1 + 3 eps + z1]) /
      (s Gamma[1 - 3 eps] Gamma[1 - 3 eps - z1]), {{eps → 0, eta → 0}, {z1 → -1/6}}]},
  MBint[((-s)^(-2 eps) Gamma[-2 eps] Gamma[-z1] Gamma[1 - eps + z1]
       Gamma[-2 eps - z1 - z2] Gamma[-eps - z1 - z2] Gamma[-z2]
       Gamma[-eps + z2] Gamma[1 + z1 + z2] Gamma[1 + 2 eps + z1 + z2]) /
     (s Gamma[1 - 3 eps] Gamma[1 - 2 eps - z1 - z2] Gamma[1 - 2 eps + z1 + z2]),
   {{eps → 0, eta → 0}, {z1 → -1/6, z2 → -1/12}}]]},
 {{{MBint[-((-s)^(-2 eps) Gamma[-2 eps] Gamma[-eps] Gamma[-3 eps - z2]
          Gamma[-2 eps - z2] Gamma[-z2] Gamma[eps + z2] Gamma[1 + 3 eps + z2]) /
        (s Gamma[-3 eps] Gamma[1 - 3 eps - z2]), {{eps → 0, eta → 0}, {z2 → -1/12}}]},
   MBint[-((-s)^(-2 eps) Gamma[-eps] Gamma[-2 eps - z2] Gamma[-z2] Gamma[-1 - 2 eps - z3]
         Gamma[-1 - 3 eps - z2 - z3] Gamma[-3 eps - z2 - z3] Gamma[-z3]
         Gamma[1 + z3]^2 Gamma[1 + eps + z2 + z3] Gamma[2 + 3 eps + z2 + z3]) /
       (s Gamma[-3 eps] Gamma[-3 eps - z2] Gamma[1 - 3 eps - z2]),
    {{eps → 0, eta → 0}, {z2 → -1/12, z3 → -101/192}}]],
  {MBint[-((-s)^(-2 eps) Gamma[-eps] Gamma[1 - 2 eps - z1] Gamma[-z1]
         Gamma[-1 - 2 eps - z3] Gamma[-1 - 3 eps - z1 - z3] Gamma[-3 eps - z1 - z3]
         Gamma[-z3] Gamma[1 + z3]^2 Gamma[1 + eps + z1 + z3] Gamma[2 + 3 eps + z1 + z3]) /
```

```
(s Gamma[1 - 3 eps] Gamma[-3 eps - z1] Gamma[1 - 3 eps - z1]),
  {{eps -> 0, eta -> 0}, {z1 -> -1/6, z3 -> -101/192}}]]},
 {{{MBint[-(((-s)^(-2 eps) Gamma[1 - 2 eps] Gamma[-2 eps]^3 Gamma[-eps]^2 Gamma[1 + 2 eps]^2) /
     (s Gamma[1 - 4 eps] Gamma[-4 eps]), {{eps -> 0, eta -> 0}, {}}]]},
   MBint[-(((-s)^(-2 eps) Gamma[-2 eps]^2 Gamma[-eps] Gamma[1 + 2 eps] Gamma[-2 eps - z2]
        Gamma[-eps - z2] Gamma[1 - eps - z2] Gamma[-z2] Gamma[-eps + z2]
        Gamma[1 + eps + z2]) / (s Gamma[-3 eps] Gamma[-3 eps - z2] Gamma[1 - 3 eps - z2]),
    {{eps -> 0, eta -> 0}, {z2 -> -1/12}}]]}, {MBint[
    -(((-s)^(-2 eps) Gamma[-2 eps]^2 Gamma[-eps] Gamma[1 + 2 eps] Gamma[1 - 2 eps - z1] Gamma[
         -eps - z1] Gamma[1 - eps - z1] Gamma[-z1] Gamma[-eps + z1] Gamma[1 + eps + z1]) /
      (s Gamma[1 - 3 eps] Gamma[-3 eps - z1] Gamma[1 - 3 eps - z1]),
    {{eps -> 0, eta -> 0}, {z1 -> -1/6}}]]},
  MBint[-(((-s)^(-2 eps) Gamma[-2 eps]^2 Gamma[1 + 2 eps] Gamma[1 - 2 eps - z1] Gamma[-z1]
       Gamma[-eps + z1] Gamma[-z1 - z2] Gamma[1 - z1 - z2] Gamma[-eps - z1 - z2]
       Gamma[-z2] Gamma[-eps + z2] Gamma[1 + z1 + z2]) / (s Gamma[1 - 3 eps]
       Gamma[-2 eps - z1] Gamma[-2 eps - z1 - z2] Gamma[1 - 2 eps - z1 - z2]),
   {{eps -> 0, eta -> 0}, {z1 -> -1/6, z2 -> -1/12}}]]},
 {{MBint[-(((-s)^(-2 eps) Gamma[-2 eps] Gamma[-eps] Gamma[-3 eps - z1]
       Gamma[1 - 2 eps - z1] Gamma[-z1] Gamma[eps + z1] Gamma[1 + 3 eps + z1]) /
     (s Gamma[1 - 3 eps] Gamma[1 - 3 eps - z1]), {{eps -> 0, eta -> 0}, {z1 -> -1/6}}]]},
  MBint[-(((-s)^(-2 eps) Gamma[-2 eps] Gamma[1 - 2 eps - z1] Gamma[-z1] Gamma[-eps + z1]
       Gamma[-2 eps - z1 - z2] Gamma[-eps - z1 - z2] Gamma[-z2] Gamma[-eps + z2]
       Gamma[z1 + z2] Gamma[1 + 2 eps + z1 + z2]) / (s Gamma[1 - 3 eps]
       Gamma[-2 eps - z1] Gamma[1 - 2 eps - z1 - z2] Gamma[-2 eps + z1 + z2]),
   {{eps -> 0, eta -> 0}, {z1 -> -1/6, z2 -> -1/12}}]]},
 MBint[-(((-s)^(-2 eps) Gamma[1 - 2 eps - z1] Gamma[-z1] Gamma[-eps + z1]
      Gamma[-eps - z1 - z2] Gamma[-z2] Gamma[-eps + z2] Gamma[-1 - 2 eps - z3]
      Gamma[-1 - 2 eps - z1 - z2 - z3] Gamma[-2 eps - z1 - z2 - z3] Gamma[-z3]
      Gamma[1 + z3]^2 Gamma[1 + z1 + z2 + z3] Gamma[2 + 2 eps + z1 + z2 + z3]) /
    (s Gamma[1 - 3 eps] Gamma[-2 eps - z1] Gamma[-2 eps - z1 - z2]
      Gamma[1 - 2 eps - z1 - z2] Gamma[-2 eps + z1 + z2]),
  {{eps -> 0, eta -> 0}, {z1 -> -1/6, z2 -> -1/12, z3 -> -101/192}}]]}}
```

```
ser = MBexpand[integrals, Exp[2 * eps * EulerGamma], {eps, 0, 0}];

MBanalytic = MBmerge[ser]
```

```
{MBint[ 1/(240 eps^4 s) (-120 + 60 eps^2 π^2 + 123 eps^4 π^4 + 120 eps^2 (-2 + eps^2 π^2) Log[-s]^2 +
     160 eps^3 Log[-s]^3 - 80 eps^4 Log[-s]^4 - 2200 eps^3 PolyGamma[2, 1] +
     40 eps Log[-s] (6 - 3 eps^2 π^2 + 110 eps^3 PolyGamma[2, 1])),
   {{eps → 0, eta → 0}, {}}], MBint[ 1/(24 eps^3 s Gamma[1 - z1])
  Gamma[-z1] Gamma[1 + z1] (24 eps^2 Gamma[-z1]^2 Gamma[z1] (1 + 2 eps EulerGamma -
        2 eps Log[-s] + 3 eps PolyGamma[0, 1 - z1] - 5 eps PolyGamma[0, -z1] +
        eps PolyGamma[0, z1] + 3 eps PolyGamma[0, 1 + z1]) - 2 Gamma[1 - z1]
      Gamma[-z1] (eps Gamma[z1] (6 + 12 eps EulerGamma + 12 eps^2 EulerGamma^2 -
            2 eps^2 π^2 - 12 eps Log[-s] - 24 eps^2 EulerGamma Log[-s] + 12 eps^2 Log[-s]^2 +
            3 eps^2 PolyGamma[0, 1 - z1]^2 + 27 eps^2 PolyGamma[0, -z1]^2 + 6 eps
             PolyGamma[0, z1] + 12 eps^2 EulerGamma PolyGamma[0, z1] - 12 eps^2 Log[-s]
             PolyGamma[0, z1] + 3 eps^2 PolyGamma[0, z1]^2 + 18 eps PolyGamma[0, 1 + z1] +
            36 eps^2 EulerGamma PolyGamma[0, 1 + z1] - 36 eps^2 Log[-s]
             PolyGamma[0, 1 + z1] + 18 eps^2 PolyGamma[0, z1] PolyGamma[0, 1 + z1] + 27
             eps^2 PolyGamma[0, 1 + z1]^2 - 18 eps PolyGamma[0, -z1] (1 + 2 eps EulerGamma -
               2 eps Log[-s] + eps PolyGamma[0, z1] + 3 eps PolyGamma[0, 1 + z1]) +
            6 eps PolyGamma[0, 1 - z1] (1 + 2 eps EulerGamma - 2 eps Log[-s] - 3 eps
                PolyGamma[0, -z1] + eps PolyGamma[0, z1] + 3 eps PolyGamma[0, 1 + z1]) -
            15 eps^2 PolyGamma[1, 1 - z1] + 27 eps^2 PolyGamma[1, -z1] +
            3 eps^2 PolyGamma[1, z1] + 27 eps^2 PolyGamma[1, 1 + z1]) -
        Gamma[1 + z1] (3 + 6 eps EulerGamma + 6 eps^2 EulerGamma^2 +
            4 eps^3 EulerGamma^3 + eps^2 π^2 + 2 eps^3 EulerGamma π^2 - 6 eps Log[-s] -
            12 eps^2 EulerGamma Log[-s] - 12 eps^3 EulerGamma^2 Log[-s] - 2 eps^3 π^2 Log[-s] +
            6 eps^2 Log[-s]^2 + 12 eps^3 EulerGamma Log[-s]^2 - 4 eps^3 Log[-s]^3 +
            6 eps^2 (1 + 2 eps EulerGamma - 2 eps Log[-s]) PolyGamma[0, 1 - z1]^2 +
            4 eps^3 PolyGamma[0, 1 - z1]^3 - 12 eps^2 (1 + 2 eps EulerGamma - 2 eps Log[-s])
             PolyGamma[1, 1 - z1] - 6 eps^2 PolyGamma[1, -z1] -
            12 eps^3 EulerGamma PolyGamma[1, -z1] + 12 eps^3 Log[-s] PolyGamma[1, -z1] +
            3 eps^2 PolyGamma[1, 1 + z1] + 6 eps^3 EulerGamma PolyGamma[1, 1 + z1] -
            6 eps^3 Log[-s] PolyGamma[1, 1 + z1] + 2 eps PolyGamma[0, 1 - z1]
             (3 + 6 eps EulerGamma + 6 eps^2 EulerGamma^2 + eps^2 π^2 - 6 eps Log[-s] - 12 eps^2
                EulerGamma Log[-s] + 6 eps^2 Log[-s]^2 - 12 eps^2 PolyGamma[1, 1 - z1] - 6
                eps^2 PolyGamma[1, -z1] + 3 eps^2 PolyGamma[1, 1 + z1]) + 9 eps^3 PolyGamma[
              2, 1] + 13 eps^3 PolyGamma[2, 1 - z1] + 9 eps^3 PolyGamma[2, -z1])) +
      Gamma[1 - z1]^2 Gamma[z1] (6 + 12 eps EulerGamma + 12 eps^2 EulerGamma^2 +
        8 eps^3 EulerGamma^3 + 2 eps^2 π^2 + 4 eps^3 EulerGamma π^2 - 12 eps Log[-s] -
        24 eps^2 EulerGamma Log[-s] - 24 eps^3 EulerGamma^2 Log[-s] -
        4 eps^3 π^2 Log[-s] + 12 eps^2 Log[-s]^2 + 24 eps^3 EulerGamma Log[-s]^2 -
        8 eps^3 Log[-s]^3 + 8 eps^3 PolyGamma[0, -z1]^3 - eps^3 PolyGamma[0, z1]^3 +
        6 eps PolyGamma[0, 1 + z1] + 12 eps^2 EulerGamma PolyGamma[0, 1 + z1] +
        12 eps^3 EulerGamma^2 PolyGamma[0, 1 + z1] +
        2 eps^3 π^2 PolyGamma[0, 1 + z1] - 12 eps^2 Log[-s] PolyGamma[0, 1 + z1] -
        24 eps^3 EulerGamma Log[-s] PolyGamma[0, 1 + z1] +
        12 eps^3 Log[-s]^2 PolyGamma[0, 1 + z1] + 3 eps^2 PolyGamma[0, 1 + z1]^2 +
        6 eps^3 EulerGamma PolyGamma[0, 1 + z1]^2 - 6 eps^3 Log[-s] PolyGamma[0, 1 + z1]^2 +
```

```
eps^3 PolyGamma[0, 1 + z1]^3 + 3 eps^2 PolyGamma[0, z1]^2
  (1 + 2 eps EulerGamma - 2 eps Log[-s] + eps PolyGamma[0, 1 + z1]) +
12 eps^2 PolyGamma[0, -z1]^2 (1 + 2 eps EulerGamma - 2 eps Log[-s] -
    eps PolyGamma[0, z1] + eps PolyGamma[0, 1 + z1]) -
12 eps^2 PolyGamma[1, 1 - z1] - 24 eps^3 EulerGamma PolyGamma[1, 1 - z1] +
24 eps^3 Log[-s] PolyGamma[1, 1 - z1] -
12 eps^3 PolyGamma[0, 1 + z1] PolyGamma[1, 1 - z1] - 24 eps^2 PolyGamma[1, -z1] -
48 eps^3 EulerGamma PolyGamma[1, -z1] + 48 eps^3 Log[-s] PolyGamma[1, -z1] -
24 eps^3 PolyGamma[0, 1 + z1] PolyGamma[1, -z1] + 3 eps^2 PolyGamma[1, z1] +
6 eps^3 EulerGamma PolyGamma[1, z1] - 6 eps^3 Log[-s] PolyGamma[1, z1] +
3 eps^3 PolyGamma[0, 1 + z1] PolyGamma[1, z1] + 3 eps^2 PolyGamma[1, 1 + z1] +
6 eps^3 EulerGamma PolyGamma[1, 1 + z1] - 6 eps^3 Log[-s] PolyGamma[1, 1 + z1] +
3 eps^3 PolyGamma[0, 1 + z1] PolyGamma[1, 1 + z1] -
eps PolyGamma[0, z1] (6 + 12 eps EulerGamma + 12 eps^2 EulerGamma^2 + 2 eps^2 π^2 -
    12 eps Log[-s] - 24 eps^2 EulerGamma Log[-s] + 12 eps^2 Log[-s]^2 +
    6 eps (1 + 2 eps EulerGamma - 2 eps Log[-s]) PolyGamma[0, 1 + z1] +
    3 eps^2 PolyGamma[0, 1 + z1]^2 - 12 eps^2 PolyGamma[1, 1 - z1] - 24 eps^2
      PolyGamma[1, -z1] + 3 eps^2 PolyGamma[1, z1] + 3 eps^2 PolyGamma[1, 1 + z1]) +
2 eps PolyGamma[0, -z1] (6 + 12 eps EulerGamma + 12 eps^2 EulerGamma^2 +
    2 eps^2 π^2 - 12 eps Log[-s] - 24 eps^2 EulerGamma Log[-s] + 12 eps^2 Log[-s]^2 +
    3 eps^2 PolyGamma[0, z1]^2 + 6 eps (1 + 2 eps EulerGamma - 2 eps Log[-s])
      PolyGamma[0, 1 + z1] + 3 eps^2 PolyGamma[0, 1 + z1]^2 - 6 eps PolyGamma[0, z1]
      (1 + 2 eps EulerGamma - 2 eps Log[-s] + eps PolyGamma[0, 1 + z1]) -
    12 eps^2 PolyGamma[1, 1 - z1] - 24 eps^2 PolyGamma[1, -z1] +
    3 eps^2 PolyGamma[1, z1] + 3 eps^2 PolyGamma[1, 1 + z1]) +
18 eps^3 PolyGamma[2, 1] + 18 eps^3 PolyGamma[2, 1 - z1] +
26 eps^3 PolyGamma[2, -z1] - eps^3 PolyGamma[2, z1] + eps^3 PolyGamma[2, 1 + z1])),
{{eps → 0, eta → 0}, {z1 → -1/6}}],
MBint[
  1/(8 eps^2 s Gamma[1 - z2]
   Gamma[-z2]^2
    Gamma[
     z2]
    Gamma[
     1 +
      z2])
   (12 eps Gamma[-z2] (1 + 2 eps EulerGamma - 2 eps Log[-s] + 3 eps PolyGamma[0, 1 - z2] -
        5 eps PolyGamma[0, -z2] + eps PolyGamma[0, z2] + 3 eps PolyGamma[0, 1 + z2]) -
     Gamma[1 - z2] (6 + 12 eps EulerGamma + 12 eps^2 EulerGamma^2 + 2 eps^2 π^2 -
        12 eps Log[-s] - 24 eps^2 EulerGamma Log[-s] + 12 eps^2 Log[-s]^2 +
        12 eps^2 PolyGamma[0, 1 - z2]^2 + 3 eps^2 PolyGamma[0, z2]^2 +
        6 eps PolyGamma[0, 1 + z2] + 12 eps^2 EulerGamma PolyGamma[0, 1 + z2] -
        12 eps^2 Log[-s] PolyGamma[0, 1 + z2] +
        3 eps^2 PolyGamma[0, 1 + z2]^2 - 6 eps PolyGamma[0, z2]
          (1 + 2 eps EulerGamma - 2 eps Log[-s] + eps PolyGamma[0, 1 + z2]) +
```

```
        12 eps PolyGamma[0, 1 - z2] (1 + 2 eps EulerGamma - 2 eps Log[-s] -
            eps PolyGamma[0, z2] + eps PolyGamma[0, 1 + z2]) -
        24 eps^2 PolyGamma[1, 1 - z2] - 12 eps^2 PolyGamma[1, -z2] +
        3 eps^2 PolyGamma[1, z2] + 3 eps^2 PolyGamma[1, 1 + z2])),
  {{eps -> 0, eta -> 0}, {z2 -> -1/12}}],
MBint[
  -1/(16 eps^2 s Gamma[1 - z1 - z2])
   Gamma[
     -z1 - z2] Gamma[
     -z2] Gamma[
     z2]
    (Gamma[1 - z1] Gamma[z1] Gamma[1 + z1 + z2] (-8 eps Gamma[-z1 - z2]
          (1 + eps EulerGamma - 2 eps Log[-s] - 2 eps PolyGamma[0, 1 - z1] + 2 eps
              PolyGamma[0, -z1] - eps PolyGamma[0, z1] - 3 eps PolyGamma[0, -z1 - z2] +
            2 eps PolyGamma[0, 1 - z1 - z2] - eps PolyGamma[0, z2] + 2 eps
              PolyGamma[0, z1 + z2] + 2 eps PolyGamma[0, 1 + z1 + z2]) + Gamma[1 - z1 - z2]
          (4 + 4 eps EulerGamma + 2 eps^2 EulerGamma^2 + eps^2 π^2 - 8 eps Log[-s] -
            8 eps^2 EulerGamma Log[-s] + 8 eps^2 Log[-s]^2 + 8 eps^2 PolyGamma[0, 1 - z1]^2 +
            8 eps^2 PolyGamma[0, -z1]^2 - 4 eps PolyGamma[0, z1] -
            4 eps^2 EulerGamma PolyGamma[0, z1] + 8 eps^2 Log[-s] PolyGamma[0, z1] +
            2 eps^2 PolyGamma[0, z1]^2 + 4 eps PolyGamma[0, -z1 - z2] + 4 eps^2 EulerGamma
              PolyGamma[0, -z1 - z2] - 8 eps^2 Log[-s] PolyGamma[0, -z1 - z2] - 4 eps^2
              PolyGamma[0, z1] PolyGamma[0, -z1 - z2] + 2 eps^2 PolyGamma[0, -z1 - z2]^2 +
            8 eps PolyGamma[0, 1 - z1 - z2] + 8 eps^2 EulerGamma PolyGamma[0, 1 - z1 - z2] -
            16 eps^2 Log[-s] PolyGamma[0, 1 - z1 - z2] - 8 eps^2 PolyGamma[0, z1]
              PolyGamma[0, 1 - z1 - z2] + 8 eps^2 PolyGamma[0, -z1 - z2] PolyGamma[0, 1 -
                z1 - z2] + 8 eps^2 PolyGamma[0, 1 - z1 - z2]^2 - 4 eps PolyGamma[0, z2] -
            4 eps^2 EulerGamma PolyGamma[0, z2] + 8 eps^2 Log[-s] PolyGamma[0, z2] +
            4 eps^2 PolyGamma[0, z1] PolyGamma[0, z2] - 4 eps^2 PolyGamma[0, -z1 - z2]
              PolyGamma[0, z2] - 8 eps^2 PolyGamma[0, 1 - z1 - z2] PolyGamma[0, z2] +
            2 eps^2 PolyGamma[0, z2]^2 + 8 eps PolyGamma[0, -z1] (1 + eps EulerGamma -
                2 eps Log[-s] - eps PolyGamma[0, z1] + eps PolyGamma[0, -z1 - z2] +
                2 eps PolyGamma[0, 1 - z1 - z2] - eps PolyGamma[0, z2]) -
            8 eps PolyGamma[0, 1 - z1] (1 + eps EulerGamma - 2 eps Log[-s] +
                2 eps PolyGamma[0, -z1] - eps PolyGamma[0, z1] + eps PolyGamma[0,
                  -z1 - z2] + 2 eps PolyGamma[0, 1 - z1 - z2] - eps PolyGamma[0, z2]) +
            8 eps^2 PolyGamma[1, 1 - z1] - 8 eps^2 PolyGamma[1, -z1] +
            2 eps^2 PolyGamma[1, z1] - 6 eps^2 PolyGamma[1, -z1 - z2] -
            8 eps^2 PolyGamma[1, 1 - z1 - z2] + 2 eps^2 PolyGamma[1, z2])) +
      Gamma[-z1] Gamma[1 + z1] (16 eps^2 Gamma[-z1 - z2] Gamma[z1 + z2] +
          Gamma[1 - z1 - z2] Gamma[1 + z1 + z2] (4 + 4 eps EulerGamma + 2 eps^2 EulerGamma^2 +
            eps^2 π^2 - 8 eps Log[-s] - 8 eps^2 EulerGamma Log[-s] + 8 eps^2 Log[-s]^2 +
            2 eps^2 PolyGamma[0, 1 + z1]^2 + 2 eps^2 PolyGamma[0, -z1 - z2]^2 +
            8 eps PolyGamma[0, 1 - z1 - z2] + 8 eps^2 EulerGamma PolyGamma[0, 1 - z1 - z2] -
            16 eps^2 Log[-s] PolyGamma[0, 1 - z1 - z2] + 8 eps^2 PolyGamma[0, 1 - z1 - z2]^2 -
```

```
          4 eps PolyGamma[0, z2] - 4 eps^2 EulerGamma PolyGamma[0, z2] +
          8 eps^2 Log[-s] PolyGamma[0, z2] - 8 eps^2 PolyGamma[0, 1 - z1 - z2]
           PolyGamma[0, z2] + 2 eps^2 PolyGamma[0, z2]^2 + 4 eps PolyGamma[0, -z1 - z2]
           (1 + eps EulerGamma - 2 eps Log[-s] + 2 eps PolyGamma[0, 1 - z1 - z2] -
             eps PolyGamma[0, z2]) - 4 eps PolyGamma[0, 1 + z1]
           (1 + eps EulerGamma - 2 eps Log[-s] + eps PolyGamma[0, -z1 - z2] +
             2 eps PolyGamma[0, 1 - z1 - z2] - eps PolyGamma[0, z2]) +
          2 eps^2 PolyGamma[1, 1 + z1] - 6 eps^2 PolyGamma[1, -z1 - z2] -
          8 eps^2 PolyGamma[1, 1 - z1 - z2] + 2 eps^2 PolyGamma[1, z2]))),
 {{eps -> 0, eta -> 0}, {z1 -> -1/6, z2 -> -1/12}}],
MBint[
 1/(eps s Gamma[1 - z1])
  Gamma[
    -z1 -
    z3]
   Gamma[-z3] Gamma[1 + z3]^2
   (Gamma[1 - z1]
      Gamma[-1 - z3]
      Gamma[-1 - z1 - z3]
      Gamma[1 + z1 + z3]
      Gamma[2 + z1 + z3]
      (1 - 2 eps Log[-s] + eps PolyGamma[0, 1 - z1] +
        3 eps PolyGamma[0, -z1] - 2 eps PolyGamma[0, -1 - z3] -
        3 eps PolyGamma[0, -1 - z1 - z3] - 3 eps PolyGamma[0, -z1 - z3] +
        eps PolyGamma[0, 1 + z1 + z3] + 3 eps PolyGamma[0, 2 + z1 + z3]) +
     Gamma[-z1] (Gamma[-z1 - z3] Gamma[-z3] Gamma[1 + z1 + z3]^2
         (-1 + 2 eps Log[-s] - 3 eps PolyGamma[0, 1 - z1] -
           eps PolyGamma[0, -z1] + 6 eps PolyGamma[0, -z1 - z3] +
           2 eps PolyGamma[0, -z3] - 4 eps PolyGamma[0, 1 + z1 + z3]) +
        Gamma[-1 - z3] Gamma[-1 - z1 - z3] Gamma[2 + z1 + z3]^2
         (1 - 2 eps Log[-s] + 3 eps PolyGamma[0, 1 - z1] + eps PolyGamma[0, -z1] -
           2 eps PolyGamma[0, -1 - z3] - 3 eps PolyGamma[0, -1 - z1 - z3] -
           3 eps PolyGamma[0, -z1 - z3] + 4 eps PolyGamma[0, 2 + z1 + z3]))),
 {{eps -> 0, eta -> 0}, {z1 -> -1/6, z3 -> -101/192}}],
MBint[
 -(3
     Gamma[-z2]
     Gamma[-1 - z3]
     Gamma[-1 - z2 - z3]
     Gamma[-z2 - z3]
     Gamma[-z3]
     Gamma[1 + z3]^2
     Gamma[1 + z2 + z3]
```

```
      Gamma[2 + z2 + z3]) /
    (s Gamma[1 - z2]), {{eps → 0,
    eta →
     0}, {z2 →
     - 1/12,
    z3 →
     - 101/192}}],
MBint[(Gamma[-z2] Gamma[z2]
      Gamma[
       -z1 - z2 - z3] Gamma[
       -z3] Gamma[1 + z3]^2
      (-Gamma[1 - z1] Gamma[z1] Gamma[1 + z1 + z2] Gamma[-1 - z3]
          Gamma[-1 - z1 - z2 - z3] Gamma[1 + z1 + z2 + z3] Gamma[2 + z1 + z2 + z3] +
        Gamma[-z1] Gamma[1 + z1] Gamma[z1 + z2]
         (Gamma[-z1 - z2 - z3] Gamma[-z3] Gamma[1 + z1 + z2 + z3]^2 -
            Gamma[-1 - z3] Gamma[-1 - z1 - z2 - z3] Gamma[2 + z1 + z2 + z3]^2))) /
    (s Gamma[1 - z1 - z2] Gamma[z1 + z2] Gamma[1 + z1 + z2]),
  {{eps →
     0, eta →
     0}, {z1 →
     - 1/6, z2 →
     - 1/12, z3 →
     - 101/192}}]}
```

**MBintegrate[MBanalytic, {s → -1}]**

```
Shifting contours...

Performing 7 lower-dimensional integrations with NIntegrate...1...2...3...4...5...6...7

Higher-dimensional integrals

Preparing MBpart1eps0 (dim 3)

Preparing MBpart2eps0 (dim 2)

Preparing MBpart3eps0 (dim 2)

Preparing MBpart4eps0 (dim 2)

Preparing MBpart5eps-1 (dim 2)

Preparing MBpart6eps-1 (dim 2)

Preparing MBpart7eps-2 (dim 2)

Running MBpart1eps0

Running MBpart2eps0

Running MBpart3eps0

Running MBpart4eps0

Running MBpart5eps-1

Running MBpart6eps-1

Running MBpart7eps-2
```

$$\left\{-25.029+\frac{0.5}{\text{eps}^4}-\frac{5.75727}{\text{eps}^2}-\frac{17.2295}{\text{eps}},\ \left\{0.00210093+\frac{1.79981\times10^{-15}}{\text{eps}^2}+\frac{0.000227337}{\text{eps}},\ 0\right\}\right\}$$

```
Quit[]
```

```
FIESTAPath = "your fiesta path"; Get[FIESTAPath <> "FIESTA4.m"];

UsingQLink = True; UsingC = True;

CurrentIntegratorSettings = {{"maxeval", "500000"}};
```

```
SDEvaluate[UF[{k1, k2}, {-k1^2, -(k1 - k2)^2, -k2^2, -(k1 - k2 + p1)^2, -(k2 + p2)^2,
   -(k1 + p1 + p2)^2, -(k2 + p1)^2}, {p1^2 → 0, p2^2 → 0, p1 * p2 → s / 2,
   s → -1}], {1, 1, 1, 1, 1, 1, -1}, 0]
```

```
FIESTA 4.1
Current integrator: vegasCuba
CurrentIntegratorSettings: {{"epsrel","1.000000E-05"},{"epsabs","1.000000E-12"},{"mineval"
Integration test passed
Starting 1 subkernels
Subkernel will be used for launching external programs, all evaluations go on main kernel.
UsingC: True
NumberOfLinks: 1
UsingQLink: True
Strategy: STRATEGY_S
Integration has to be performed up to order 0
KLink created (2013 version)! You can read information on QOpen, QRead, QRemoveDatabase, Q(
Sector decomposition - 6 sectors
Primary sector 1 resulted in 14 sectors.
Primary sector 2 resulted in 14 sectors.
Primary sector 3 resulted in 14 sectors.
Primary sector 4 resulted in 14 sectors.
Primary sector 5 resulted in 14 sectors.
Primary sector 6 resulted in 14 sectors.
Totally: 0.3216 seconds; 12 sectors.
Preparing database: 0.0316 seconds.
Variable substitution..........0.4767 seconds; 168 terms.
Pole resolution..........0.6253 seconds; 498 terms.
Expression preparation..........0.6127 seconds; 498 terms.
Epsilon expansion..........1.2281 seconds; 1052 terms.
Preparing integration strings..........3.2871 seconds; 978 terms.
Database ready for integration.
Terms of order -4: 8, max vars: 1
Integrating..........0.0018 seconds.
Returned answer: -0.5 + pm* 2.*^-6
(-0.5 + 2.*^-6*pm1)*ep^(-4)
Terms of order -3: 28, max vars: 2
Integrating..........0.001 seconds.
Returned answer: -3.*^-6 + pm* 0.00006
(-0.5 + 2.*^-6*pm2)*ep^(-4)+(0)*ep^(-3)
Terms of order -2: 54, max vars: 3
Integrating..........0.0008 seconds.
Returned answer: 7.402217 + pm* 0.001278
(-0.5 + 2.*^-6*pm4)*ep^(-4)+(0)*ep^(-3)+(5.757283 + 0.001278*pm6)*ep^(-2)
Terms of order -1: 100, max vars: 4
Integrating..........0.0008 seconds.
Returned answer: 15.627275 + pm* 0.005883
(-0.5 + 2.*^-6*pm7)*ep^(-4)+(0)*ep^(-3)+(5.757283 + 0.001278*pm9)*ep^(-2)+(17.230007 + 0.0(
Terms of order 0: 168, max vars: 5
Integrating..........0.0007 seconds.
Returned answer: 5.545874 + pm* 0.023509
(-0.5 + 2.*^-6*pm11)*ep^(-4)+(0)*ep^(-3)+(5.757283 + 0.001278*pm13)*ep^(-2)+(17.230007 + 0
Total integration time: 95.7775
Total time used: 102.579 seconds.
```

$$25.0277 + \frac{-0.5 + 2.\times 10^{-6}\,\text{pm16}}{\text{ep}^4} + \frac{5.75728 + 0.001278\,\text{pm18}}{\text{ep}^2} + \frac{17.23 + 0.005886\,\text{pm19}}{\text{ep}} + 0.023882\,\text{pm20}$$

```
Quit[]
```